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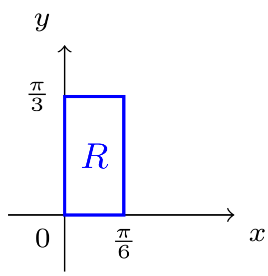
UNIVERSIDADE FEDERAL DA GRANDE DOURADOS
Cálculo de Várias Variáveis — Avaliação P2
Prof. Adriano Barbosa

Matemática 21/02/2024

Aluno(a):..

Todas as respostas devem ser justificadas.

1. (2 pts) Calcule a integral $\iint_R x \operatorname{sen}(x+y)\, dA$, onde R é a região da figura abaixo.



2. (2 pts) Calcule a integral dupla $\int_0^2 \int_{-y}^{2y} xe^{y^3}\, dxdy$.

3. (2 pts) Use coordenadas polares para determinar o volume do sólido acima do cone $z = -\sqrt{x^2+y^2}$ e abaixo do disco $x^2 + y^2 \leq 4$.

4. (2 pts) Calcule $\iiint_E y\, dV$, onde E é a região do espaço limitada pelos planos $x = 0$, $y = 0$, $z = 0$ e $x + y + z = 1$.

5. Seja E a região limitada acima pela esfera $\rho = a$ e abaixo pelo cone $\phi = \frac{\pi}{3}$. Expresse a integral $\iiint_E x^2 + y^2\, dV$ em:

   (a) (2/3 pt) Coordenadas esféricas.

   (b) (2/3 pt) Coordenadas cilíndricas.

   (c) (2/3 pt) Coordenadas cartesianas.

Boa Prova!

Avaliação P2

① $$\iint_R x\,\text{sen}(x+y)\,dA = \int_0^{\pi/3}\int_0^{\pi/6} x\,\text{sen}(x+y)\,dx\,dy$$

Integrando por partes:

$$u = x \qquad\qquad du = dx$$
$$\Rightarrow$$
$$dv = \text{sen}(x+y)\,dx \qquad\qquad v = -\cos(x+y)$$

$$\therefore \int_0^{\pi/6} x\,\text{sen}(x+y)\,dx = -x\cos(x+y)\Big|_{x=0}^{x=\pi/6} + \int_0^{\pi/6}\cos(x+y)\,dx$$

$$= -\frac{\pi}{6}\cos\left(\frac{\pi}{6}+y\right) + \text{sen}(x+y)\Big|_{x=0}^{x=\pi/6} = -\frac{\pi}{6}\cos\left(\frac{\pi}{6}+y\right) + \text{sen}\left(\frac{\pi}{6}+y\right) - \text{sen}(y)$$

Logo,

$$\iint_R x\,\text{sen}(x+y)\,dA = \int_0^{\pi/3} -\frac{\pi}{6}\cos\left(\frac{\pi}{6}+y\right) + \text{sen}\left(\frac{\pi}{6}+y\right) - \text{sen}(y)\,dy$$

$$= -\frac{\pi}{6}\,\text{sen}\left(\frac{\pi}{6}+y\right)\Big|_0^{\pi/3} - \cos\left(\frac{\pi}{6}+y\right)\Big|_0^{\pi/3} + \cos y\Big|_0^{\pi/3}$$

$$= -\frac{\pi}{6}\left[\text{sen}\left(\frac{\pi}{6}+\frac{\pi}{3}\right) - \text{sen}\left(\frac{\pi}{6}\right)\right] - \cos\left(\frac{\pi}{6}+\frac{\pi}{3}\right) + \cos\left(\frac{\pi}{6}\right) + \cos\frac{\pi}{3} - \cos 0$$

$$= -\frac{\pi}{6}\left[\text{sen}\frac{\pi}{2} - \text{sen}\frac{\pi}{6}\right] - \cos\frac{\pi}{2} + \cos\frac{\pi}{6} + \cos\frac{\pi}{3} - \cos 0$$

$$= -\frac{\pi}{6}\left[1 - \frac{1}{2}\right] - 0 + \frac{\sqrt{3}}{2} + \frac{1}{2} - 1 = -\frac{\pi}{12} + \frac{\sqrt{3}}{2} - \frac{1}{2} = \frac{-\pi + 6\sqrt{3} - 6}{12}$$

(2) $$\int_0^2\int_{-y}^{2y} x e^{y^3}\,dx\,dy = \int_0^2 e^{y^3}\left(\frac{x^2}{2}\Big|_{-y}^{2y}\right)dy = \int_0^2 e^{y^3}\left(2y^2-\frac{y^2}{2}\right)dy$$

$$= \frac{3}{2}\int_0^2 y^2 e^{y^3}\,dy .$$

$u = y^3 \Rightarrow du = 3y^2 dy \Rightarrow y^2 dy = \frac{1}{3}du$

$y=0 \Rightarrow u=0$ e $y=2 \Rightarrow u=8$

Aplicando a mudança de variáveis:

$$\int_0^2\int_{-y}^{2y} x e^{y^3}\,dx\,dy = \frac{3}{2}\int_0^8 e^u \cdot \frac{1}{3}du = \frac{1}{2}\left(e^u\Big|_0^8\right) = \frac{1}{2}\left(e^8-e^0\right) = \frac{1}{2}\left(e^8-1\right)$$

(3) Como o cone está abaixo do plano xy $\left(z=-\sqrt{x^2+y^2}\leq 0\right)$, o volume do sólido é dado por

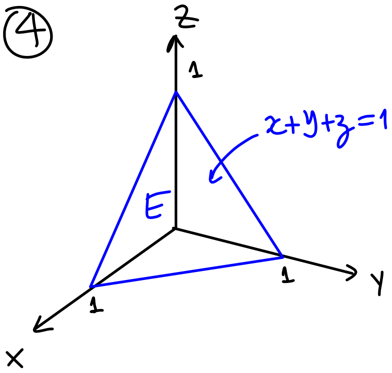
$$V = -\iint_R -\sqrt{x^2+y^2}\,dA,$$ onde R é o disco $x^2+y^2\leq 4$.

Em coord. polares: $R=\{(r,\theta)\mid 0\leq r\leq 2,\ 0\leq\theta\leq 2\pi\}$

$$\Rightarrow V = -\int_0^{2\pi}\int_0^2 -\sqrt{r^2}\cdot r\,dr\,d\theta = \int_0^{2\pi}\int_0^2 r^2\,dr\,d\theta = \int_0^{2\pi}d\theta\cdot\int_0^2 r^2\,dr$$

$$= 2\pi\cdot\left(\frac{r^3}{3}\Big|_0^2\right) = 2\pi\cdot\frac{8}{3} = \frac{16\pi}{3}.$$

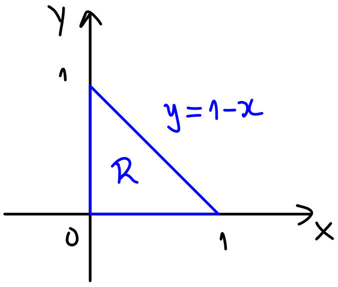
④



$$R=\{(x,y)\mid 0\le x\le 1,\ 0\le y\le 1-x\}$$

$$E=\{(x,y,z)\mid (x,y)\in R,\ 0\le z\le 1-x-y\}$$

$$=\{(x,y,z)\mid 0\le x\le 1,\ 0\le y\le 1-x,\ 0\le z\le 1-x-y\}$$



$$\therefore \iiint_E y\,dV=\int_0^1\int_0^{1-x}\int_0^{1-x-y} y\ dz\,dy\,dx$$

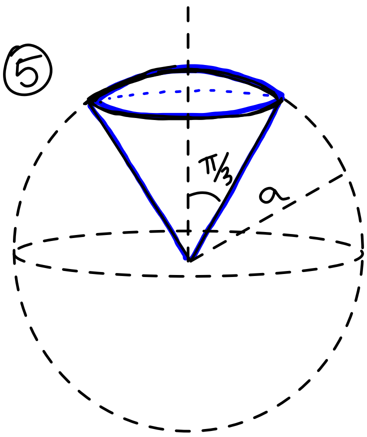
$$=\int_0^1\int_0^{1-x} y\left(z\Big|_0^{1-x-y}\right)dy\,dx=\int_0^1\int_0^{1-x} y\,(1-x-y)\,dy\,dx$$

$$=\int_0^1\int_0^{1-x} y(1-x)-y^2\,dy\,dx=\int_0^1 (1-x)\frac{y^2}{2}-\frac{y^3}{3}\Big|_{y=0}^{y=1-x}dx$$

$$=\int_0^1\frac{(1-x)(1-x)^2}{2}-\frac{(1-x)^3}{3}\,dx=\int_0^1\frac{(1-x)^3}{2}-\frac{(1-x)^3}{3}\,dx=\int_0^1\frac{3(1-x)^3-2(1-x)^3}{6}\,dx$$

$$=\int_0^1\frac{(1-x)^3}{6}\,dx=\frac{1}{6}\left[-\frac{(1-x)^4}{4}\Big|_0^1\right]=\frac{1}{6}\left[-\frac{(1-1)^4}{4}+\frac{(1-0)^4}{4}\right]=\frac{1}{6}\cdot\frac{1}{4}=\frac{1}{24}$$

⑤



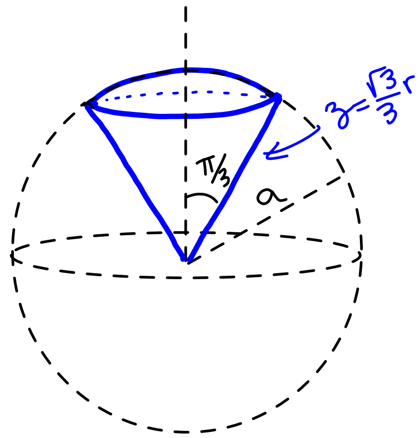
A região de integração é uma seção cônica de uma esfera de raio a.

a) $E=\{(\rho,\theta,\phi)\mid 0\le\rho\le a,\ 0\le\theta\le 2\pi,\ 0\le\phi\le\frac{\pi}{3}\}$

$$\iiint_E x^2+y^2\,dV=\int_0^a\int_0^{2\pi}\int_0^{\pi/3}\left(\rho^2\sin^2\phi\cos^2\theta+\rho^2\sin^2\phi\sin^2\theta\right)\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$

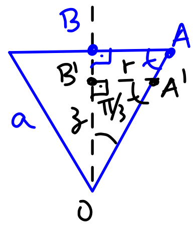
$$=\int_0^a\int_0^{2\pi}\int_0^{\pi/3}\rho^2\sin^2\phi\left(\cos^2\theta+\sin^2\theta\right)\rho^2\sin\phi\,d\rho\,d\theta\,d\phi=\int_0^a\int_0^{2\pi}\int_0^{\pi/3}\rho^4\sin^3\phi\,d\rho\,d\theta\,d\phi.$$

b)



A esfera tem eq. cartesiana $x^2+y^2+z^2=a^2$, logo a tampa do cone pode ser escrita como $z=\sqrt{a^2-x^2-y^2} \Rightarrow z=\sqrt{a^2-r^2}$.

Cortando o cone com o plano xz, temos



que os triângulos OAB e OA'B' são semelhantes pelo caso AAA. Além disso,

$$\cos\frac{\pi}{3}=\frac{\overline{OB}}{a} \Rightarrow \overline{OB}=\frac{a}{2}.$$ Por Pitágoras em OAB:

$$a^2=\overline{OB}^2+\overline{AB}^2 \Rightarrow \overline{AB}^2=a^2-\frac{a^2}{4}=\frac{3}{4}a^2 \Rightarrow \overline{AB}=\frac{a}{2}\sqrt{3}.$$

Pela semelhança,

$$\frac{z}{r}=\frac{\frac{a}{2}}{\frac{a}{2}\sqrt{3}} \Rightarrow z=\frac{1}{\sqrt{3}}r \Rightarrow z=\frac{\sqrt{3}}{3}r$$

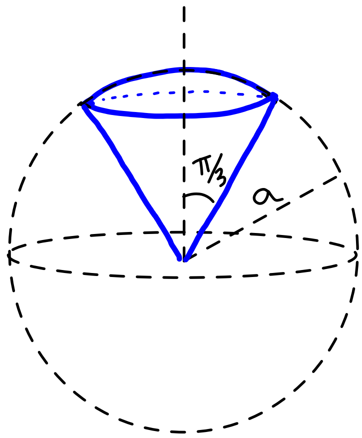
Portanto,

$$E=\left\{(r,\theta,z)\ \middle|\ 0\le r\le \frac{a}{2}\sqrt{3},\ 0\le\theta\le 2\pi,\ \frac{\sqrt{3}}{3}r\le z\le\sqrt{a^2-r^2}\right.$$

$$\iiint_E x^2+y^2\,dV=\int_0^{\frac{a}{2}\sqrt{3}}\int_0^{2\pi}\int_{\frac{\sqrt{3}}{3}r}^{\sqrt{a^2-r^2}}\left(r^2\cos^2\theta+r^2\sin^2\theta\right)r\,dz\,d\theta\,dr$$

$$=\int_0^{\frac{a}{2}\sqrt{3}}\int_0^{2\pi}\int_{\frac{\sqrt{3}}{3}r}^{\sqrt{a^2-r^2}}r^3\,dz\,d\theta\,dr$$

c)



$$x^2+y^2+z^2=a^2 \Rightarrow z=\sqrt{a^2-x^2-y^2}$$

$$z=\frac{\sqrt{3}}{3}r=\frac{\sqrt{3}}{3}\sqrt{x^2+y^2} \Rightarrow z=\frac{\sqrt{3(x^2+y^2)}}{3}$$

são as eq. cartesianas da esfera e do cone como funções de x e y. Além disso, a boca do cone tem equação

$$x^2+y^2=\left(\frac{a}{2}\sqrt{3}\right)^2 \text{ e } z=\frac{a}{2} \Rightarrow y=\pm\sqrt{\left(\frac{a}{2}\sqrt{3}\right)^2-x^2} \text{ e } z=\frac{a}{2}.$$

Logo,

$$E=\left\{(x,y,z) \,\middle|\, -\frac{a}{2}\sqrt{3}\le x\le \frac{a}{2}\sqrt{3},\ -\sqrt{\left(\frac{a}{2}\sqrt{3}\right)^2-x^2}\le y\le \sqrt{\left(\frac{a}{2}\sqrt{3}\right)^2-x^2},\ \frac{\sqrt{3(x^2+y^2)}}{3}\le z\le \sqrt{a^2-x^2-y^2}\right\}$$

$$\therefore \iiint_E x^2+y^2\,dV=\int_{-\frac{a}{2}\sqrt{3}}^{\frac{a}{2}\sqrt{3}}\int_{-\sqrt{\left(\frac{a}{2}\sqrt{3}\right)^2-x^2}}^{\sqrt{\left(\frac{a}{2}\sqrt{3}\right)^2-x^2}}\int_{\frac{\sqrt{3(x^2+y^2)}}{3}}^{\sqrt{a^2-x^2-y^2}} x^2+y^2\,dz\,dy\,dx$$