1	
2	
3	
4	
5	
Nota	

UNIVERSIDADE FEDERAL DA GRANDE DOURADOS
Cálculo de Várias Variáveis — Avaliação PS
Prof. Adriano Barbosa

Matemática 06/09/2023

Aluno(a):..

Todas as respostas devem ser justificadas.

Avaliação P1:

1. Esboce o maior domínio das funções:

   (a) $f(x, y) = \ln(1 - x^2 - y^2)$

   (b) $f(x, y) = \dfrac{1}{x - y^2}$

2. Suponha $z = f(x, y)$, onde $x = g(s, t)$, $y = h(s, t)$, $g(1, 2) = 3$, $\dfrac{\partial g}{\partial s}(1, 2) = -1$, $\dfrac{\partial g}{\partial t}(1, 2) = 4$, $h(1, 2) = 6$, $\dfrac{\partial h}{\partial s}(1, 2) = -5$, $\dfrac{\partial h}{\partial t}(1, 2) = 10$, $\dfrac{\partial f}{\partial x}(3, 6) = 7$ e $\dfrac{\partial f}{\partial y}(3, 6) = 8$. Determine o valor de $\dfrac{\partial z}{\partial s}$ e $\dfrac{\partial z}{\partial t}$ quando $s = 1$ e $t = 2$.

3. A temperatura T de um ponto P numa bola de metal é inversamente proporcional à distância de P ao centro da bola, que tomamos como sendo a origem. A temperatura no ponto $(1, 2, 2)$ é de 120°C. Determine a taxa de variação de T em $(1, 2, 2)$ na direção $(1, -1, 1)$.

4. Determine se as afirmações são verdadeiras ou falsas justificando.

   (a) Se f tem um mínimo local em (a, b) e f é diferenciável em (a, b), então $\nabla f(a, b) = (0, 0)$.

   (b) Se $f(x, y) = \ln y$, então $\nabla f(x, y) = \dfrac{1}{y}$.

5. Determine os máximos e mínimos de $f(x, y, z) = 2x + 2y + z$ restrita a $x^2 + y^2 + z^2 = 9$.

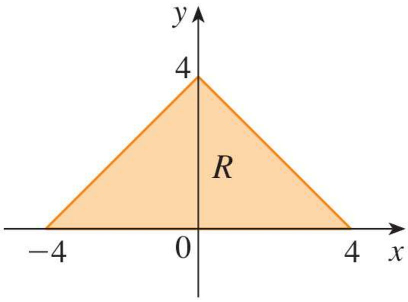
Avaliação P2:

1. Calcule a integral $\displaystyle\iint_R x \operatorname{sen}(x + y)\, dA$, onde $R = [0, \pi/6] \times [0, \pi/3]$.

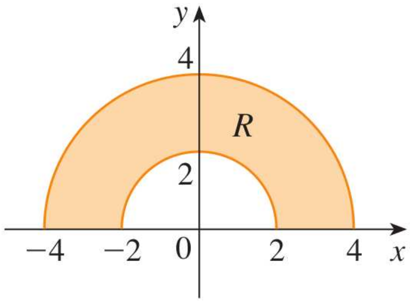
2. Calcule a integral iterada invertendo a ordem de integração $\displaystyle\int_0^1 \int_x^1 \cos(y^2)\, dy\, dx$.

3. Escreva $\iint_R f(x,y)\, dA$ como uma integral iterada para cada uma das regiões R abaixo.

(a)

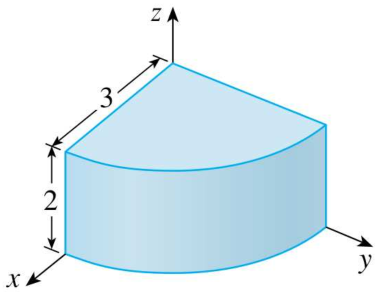


(b)



4. (a) Escreva a integral tripla de uma função contínua $f(x,y,z)$ sobre o sólido abaixo determinando seus limites de integração.

   (b) Calcule o volume do sólido utilizando a integral tripla encontrada no item anterior.

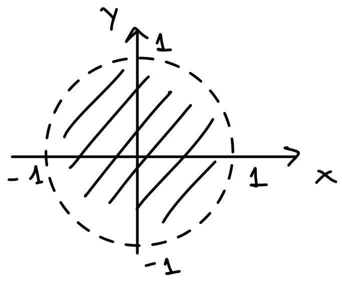


5. Calcule $\iiint_E x^2 + y^2\, dV$, onde E está entre as esferas $x^2+y^2+z^2 = 4$ e $x^2+y^2+z^2 = 9$.

Boa Prova!

Avaliação P1

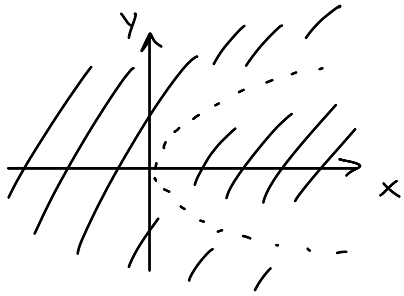
1. a) Precisamos que



$$1-x^2-y^2>0 \Rightarrow x^2+y^2<1$$

(círculo)

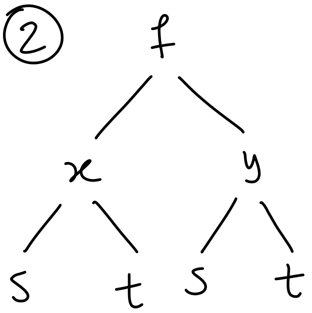
b) Devemos ter



$$x-y^2 \neq 0 \Rightarrow x \neq y^2$$

(parábola)

2.



Pela regra da cadeia:

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s}$$

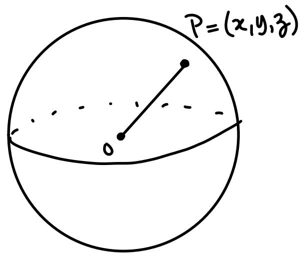
$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t}$$

Quando $s=1$ e $t=2$, temos $x=g(1,2)=3$, $y=h(1,2)=6$ e

$$\frac{\partial z}{\partial s} = 7\cdot(-1) + 8\cdot(-5) = -47$$

$$\frac{\partial z}{\partial t} = 7\cdot 4 + 8\cdot 10 = 108.$$

(3)



$$T(P) = \frac{k}{d(0,P)} \Rightarrow T(x,y,z) = \frac{k}{\sqrt{x^2+y^2+z^2}}$$

$$120 = T(1,2,2) = \frac{k}{\sqrt{1^2+2^2+2^2}} \Rightarrow k = 360.$$

A função de temperatura é $T(x,y,z) = \dfrac{360}{\sqrt{x^2+y^2+z^2}}$.

$$\Rightarrow \nabla T = \left(\frac{-360 \cdot \frac{2x}{2\sqrt{x^2+y^2+z^2}}}{\left(\sqrt{x^2+y^2+z^2}\right)^2}, \frac{-360 \cdot \frac{2y}{2\sqrt{x^2+y^2+z^2}}}{\left(\sqrt{x^2+y^2+z^2}\right)^2}, \frac{-360 \cdot \frac{2z}{2\sqrt{x^2+y^2+z^2}}}{\left(\sqrt{x^2+y^2+z^2}\right)^2} \right)$$

$$= \left(\frac{-360x}{(x^2+y^2+z^2)^{3/2}}, \frac{-360y}{(x^2+y^2+z^2)^{3/2}}, \frac{-360z}{(x^2+y^2+z^2)^{3/2}} \right)$$

Tomando a direção unitária:

$$\|(1,-1,1)\| = \sqrt{1^2+(-1)^2+1^2} = \sqrt{3} \Rightarrow u = \frac{1}{\sqrt{3}}(1,-1,1) = \frac{\sqrt{3}}{3}(1,-1,1)$$

Portanto,

$$\frac{\partial T}{\partial u}(1,2,2) = \nabla T(1,2,2) \cdot u = \left(\frac{-360}{27}, \frac{-720}{27}, \frac{-720}{27} \right) \cdot \frac{\sqrt{3}}{3}(1,-1,1)$$

$$= \frac{\sqrt{3}}{3}\left(-\frac{360}{27} + \frac{720}{27} - \frac{720}{27} \right) = -\frac{360\sqrt{3}}{81} = -\frac{40\sqrt{3}}{9}.$$

(4) a) Se f é diferenciável e tem um mínimo local em (a,b), então $\frac{\partial f}{\partial x}(a,b) = \frac{\partial f}{\partial y}(a,b) = 0$. Logo, $\nabla f(a,b) = \left(\frac{\partial f}{\partial x}(a,b), \frac{\partial f}{\partial y}(a,b)\right) = (0,0)$.

b) Se $f(x,y) = \ln y$, então $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = \left(0, \frac{1}{y}\right)$.

⑤ Aplicando o método dos multiplicadores de Lagrange:

$f(x,y,z) = 2x + 2y + z \Rightarrow \nabla f = (2,2,1)$

$g(x,y,z) = x^2 + y^2 + z^2 \Rightarrow \nabla g = (2x, 2y, 2z)$

$$\therefore \begin{cases} \nabla f = \lambda \nabla g \\ g(x,y,z) = 9 \end{cases} \Rightarrow \begin{cases} 2 = 2\lambda x & (\div 2) \\ 2 = 2\lambda y & (\div 2) \\ 1 = 2\lambda z & (\div 2) \\ x^2+y^2+z^2 = 9 \end{cases} \Rightarrow \begin{cases} \lambda x = 1 \\ \lambda y = 1 \\ \lambda z = 1/2 \\ x^2+y^2+z^2 = 9 \end{cases}$$

Note que $\lambda \neq 0$, caso contrário, por ①, teríamos $0 = 1$.

$$\therefore \begin{cases} x = 1/\lambda & ① \\ y = 1/\lambda & ② \\ z = 1/2\lambda & ③ \\ x^2+y^2+z^2 = 9 & ④ \end{cases}$$

Substituindo ①, ② e ③ em ④:

$$\left(\frac{1}{\lambda}\right)^2 + \left(\frac{1}{\lambda}\right)^2 + \left(\frac{1}{2\lambda}\right)^2 = 9 \Rightarrow \frac{1}{\lambda^2} + \frac{1}{\lambda^2} + \frac{1}{4\lambda^2} = 9 \Rightarrow \frac{4+4+1}{4\lambda^2} = 9$$

$$\Rightarrow 36\lambda^2 = 9 \Rightarrow \lambda^2 = \frac{1}{4} \Rightarrow \lambda = \pm\frac{1}{2}$$

P/ $\lambda = \frac{1}{2}$: $x = 2$, $y = 2$, $z = 1$

P/ $\lambda = -\frac{1}{2}$: $x = -2$, $y = -2$, $z = -1$.

Avaliando f nos pontos:

$f(2,2,1) = 4 + 4 + 1 = 9$

$f(-2,-2,-1) = -4 - 4 - 1 = -9$

Assim, $(2,2,1)$ é ponto de máximo e $(-2,-2,-1)$ é ponto de mínimo.

Avaliação P2

① $I = \iint_R x \operatorname{sen}(x+y)\, dA = \int_0^{\pi/3} \int_0^{\pi/6} x \operatorname{sen}(x+y)\, dx\, dy$

Integrando por partes:

$$u = x \qquad\qquad du = dx$$
$$\Rightarrow$$
$$dv = \operatorname{sen}(x+y)\, dx \qquad\qquad v = -\cos(x+y)$$

$$\therefore \int_0^{\pi/6} x \operatorname{sen}(x+y)\, dx = -x\cos(x+y)\Big|_0^{\pi/6} + \int_0^{\pi/6} \cos(x+y)\, dx$$

$$= -\frac{\pi}{6}\cos\left(\frac{\pi}{6}+y\right) + \operatorname{sen}(x+y)\Big|_0^{\pi/6} = -\frac{\pi}{6}\cos\left(\frac{\pi}{6}+y\right) + \operatorname{sen}\left(\frac{\pi}{6}+y\right) - \operatorname{sen} y$$

Logo,

$$I = \int_0^{\pi/3} -\frac{\pi}{6}\cos\left(\frac{\pi}{6}+y\right) + \operatorname{sen}\left(\frac{\pi}{6}+y\right) - \operatorname{sen} y\, dy$$

$$= -\frac{\pi}{6}\int_0^{\pi/3} \cos\left(\frac{\pi}{6}+y\right) dy + \int_0^{\pi/3} \operatorname{sen}\left(\frac{\pi}{6}+y\right) dy - \int_0^{\pi/3} \operatorname{sen} y\, dy$$

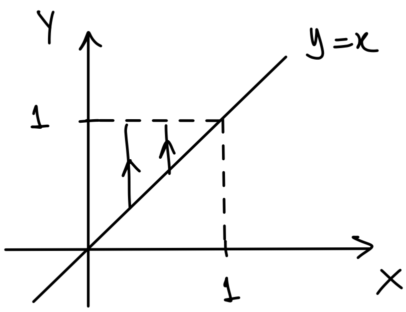
$$= -\frac{\pi}{6}\left[\operatorname{sen}\left(\frac{\pi}{6}+y\right)\Big|_0^{\pi/3}\right] + \left[-\cos\left(\frac{\pi}{6}+y\right)\Big|_0^{\pi/3}\right] + \cos y\Big|_0^{\pi/3}$$

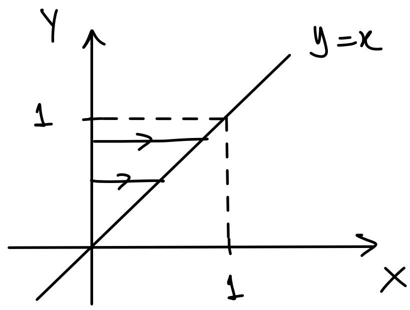
$$= -\frac{\pi}{6}\left[\operatorname{sen}\left(\frac{\pi}{6}+\frac{\pi}{3}\right) - \operatorname{sen}\left(\frac{\pi}{6}\right)\right] - \cos\left(\frac{\pi}{6}+\frac{\pi}{3}\right) + \cos\left(\frac{\pi}{6}\right) + \cos\left(\frac{\pi}{3}\right) - \cos 0$$

$$= -\frac{\pi}{6}\left[\operatorname{sen}\left(\frac{\pi}{2}\right) - \frac{1}{2}\right] - \cancelto{0}{\cos\left(\frac{\pi}{2}\right)} + \frac{\sqrt{3}}{2} + \frac{1}{2} - 1$$

$$= -\frac{\pi}{6}\left[1 - \frac{1}{2}\right] + \frac{\sqrt{3}-1}{2} = -\frac{\pi}{6}\cdot\frac{1}{2} + \frac{\sqrt{3}-1}{2} = \frac{-\pi}{12} + \frac{\sqrt{3}-1}{2} = \frac{-\pi + 6\sqrt{3} - 6}{12}$$

② $R = \{(x,y) \mid 0 \le x \le 1,\ x \le y \le 1\}$





Invertendo a ordem de integração

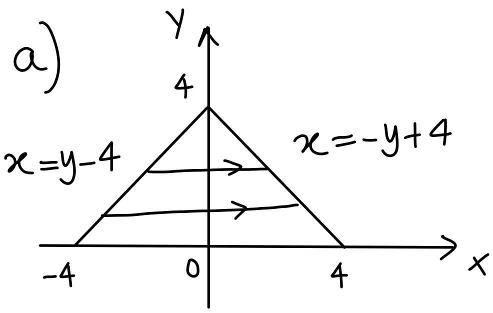
$$R = \{(x,y) \mid 0 \le y \le 1,\ 0 \le x \le y\}$$

$$\therefore \int_0^1 \int_x^1 \cos(y^2)\,dy\,dx = \int_0^1 \int_0^y \cos(y^2)\,dx\,dy = \int_0^1 \cos(y^2) \cdot \left(x\Big|_0^y\right) dy$$

$$= \int_0^1 \cos(y^2) \cdot y\,dy \qquad \left(\begin{array}{l} u = y^2 \\ du = 2y\,dy \Rightarrow y\,dy = \frac{1}{2}du \end{array} \ \Bigg|\ \begin{array}{l} y=0 \Rightarrow u=0 \\ y=1 \Rightarrow u=1 \end{array} \right)$$

$$= \int_0^1 \cos u \cdot \frac{1}{2}\,du = \frac{1}{2}\int_0^1 \cos u\,du = \frac{1}{2}\left(\operatorname{sen} u\Big|_0^1\right) = \frac{1}{2} \cdot \operatorname{sen} 1.$$

③ a)



As retas têm eq.:

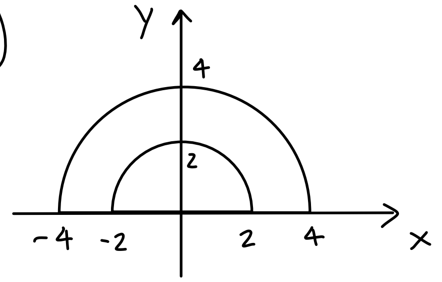
$$y = -x+4 \Leftrightarrow x = -y+4$$

$$\text{e} \quad y = x+4 \Leftrightarrow x = y-4$$

$$\therefore R = \{(x,y) \mid 0 \le y \le 4,\ y-4 \le x \le -y+4\}$$

$$\Rightarrow \iint_R f(x,y)\,dA = \int_0^4 \int_{y-4}^{-y+4} f(x,y)\,dx\,dy$$

b)



Em coord. polares:

$$R = \{(r,\theta) \mid 2 \leq r \leq 4,\ 0 \leq \theta \leq \pi\}$$

$$\therefore \iint_R f(x,y)\,dA = \int_2^4 \int_0^\pi f(r\cos\theta, r\,\text{sen}\,\theta) \cdot r\,d\theta\,dr.$$

④ a) O sólido pode ser descrito em coord. cilíndricas como

$$E = \{(r,\theta,z) \mid 0 \leq r \leq 3,\ 0 \leq \theta \leq \tfrac{\pi}{2},\ 0 \leq z \leq 2\}.$$

Assim,

$$\iiint_E f(x,y,z)\,dV = \int_0^2 \int_0^{\pi/2} \int_0^3 f(r\cos\theta, r\,\text{sen}\,\theta, z) \cdot r\,dr\,d\theta\,dz$$

b) $$V = \int_0^2 \int_0^{\pi/2} \int_0^3 r\,dr\,d\theta\,dz = \int_0^2 dz \cdot \int_0^{\pi/2} d\theta \cdot \int_0^3 r\,dr$$

$$= 2 \cdot \frac{\pi}{2} \cdot \left(\frac{r^2}{2} \Big|_0^3 \right) = \frac{9}{2}\pi.$$

⑤ A região de integração pode ser descrita em coord. esféricas como

$$E = \{(\rho, \theta, \phi) \mid 2 \le \rho \le 3,\ 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi\}.$$

Assim,

$$\iiint_E x^2+y^2 \, dV = \int_0^{\pi}\int_0^{2\pi}\int_2^3 (\rho^2 \operatorname{sen}^2\phi \cos^2\theta + \rho^2 \operatorname{sen}^2\phi \operatorname{sen}^2\theta)\,\rho^2 \operatorname{sen}\phi \, d\rho\, d\theta\, d\phi$$

$$= \int_0^{\pi}\int_0^{2\pi}\int_2^3 \rho^4 \operatorname{sen}^3\phi \, d\rho\, d\theta\, d\phi = \int_0^{2\pi} d\theta \cdot \int_2^3 \rho^4 d\rho \cdot \int_0^{\pi} \operatorname{sen}^3\phi \, d\phi$$

$$= 2\pi \cdot \left(\frac{\rho^5}{5} \Big|_2^3 \right) \cdot \int_0^{\pi} \operatorname{sen}\phi \cdot \operatorname{sen}^2\phi \, d\phi = \frac{422}{5}\pi \cdot \int_0^{\pi} \operatorname{sen}\phi\,(1-\cos^2\phi)\, d\phi$$

$$(u = \cos\phi \Rightarrow du = -\operatorname{sen}\phi\, d\phi)$$

$$= \frac{422}{5}\pi \left(-\int_1^{-1} 1-u^2 \, du \right) = \frac{422}{5}\pi \cdot \int_{-1}^{1} 1-u^2\, du = \frac{422}{5}\pi \left(u - \frac{u^3}{3} \Big|_{-1}^{1} \right)$$

$$= \frac{422}{5}\pi \left[1 - \frac{1}{3} - (-1) + \frac{(-1)^3}{3} \right] = \frac{422}{5}\pi \cdot \left(2 - \frac{2}{3} \right) = \frac{422}{5}\pi \cdot \frac{4}{3} = \frac{1688}{15}\pi.$$