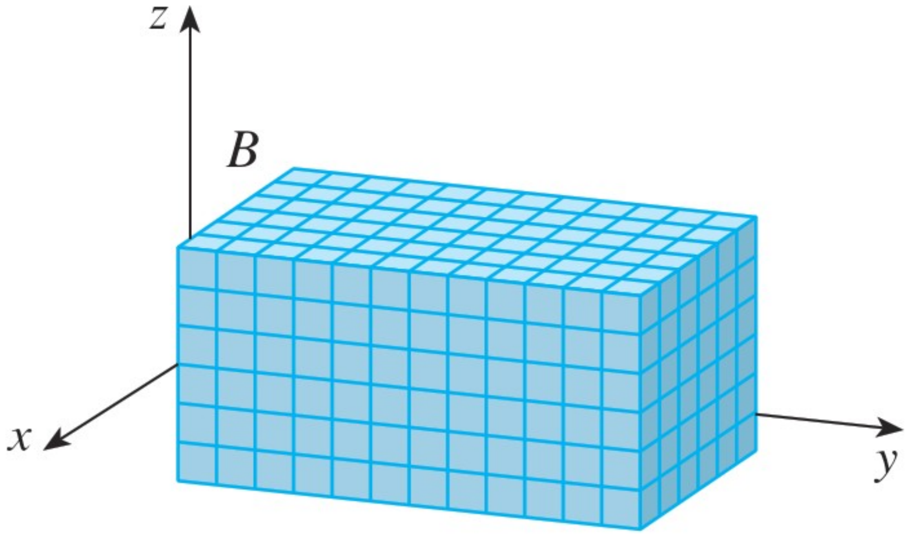
Cálculo III

Integral tripla

Prof. Adriano Barbosa

Integral tripla

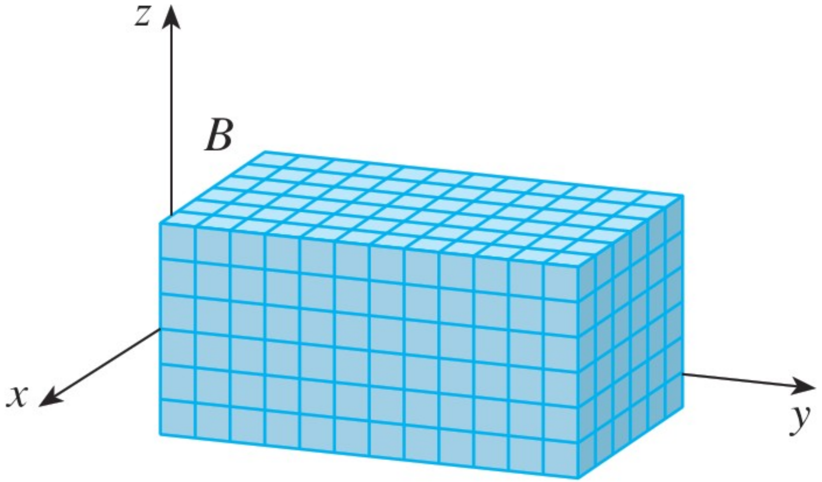
$f: B \subset \mathbb{R}^3 \rightarrow \mathbb{R}$

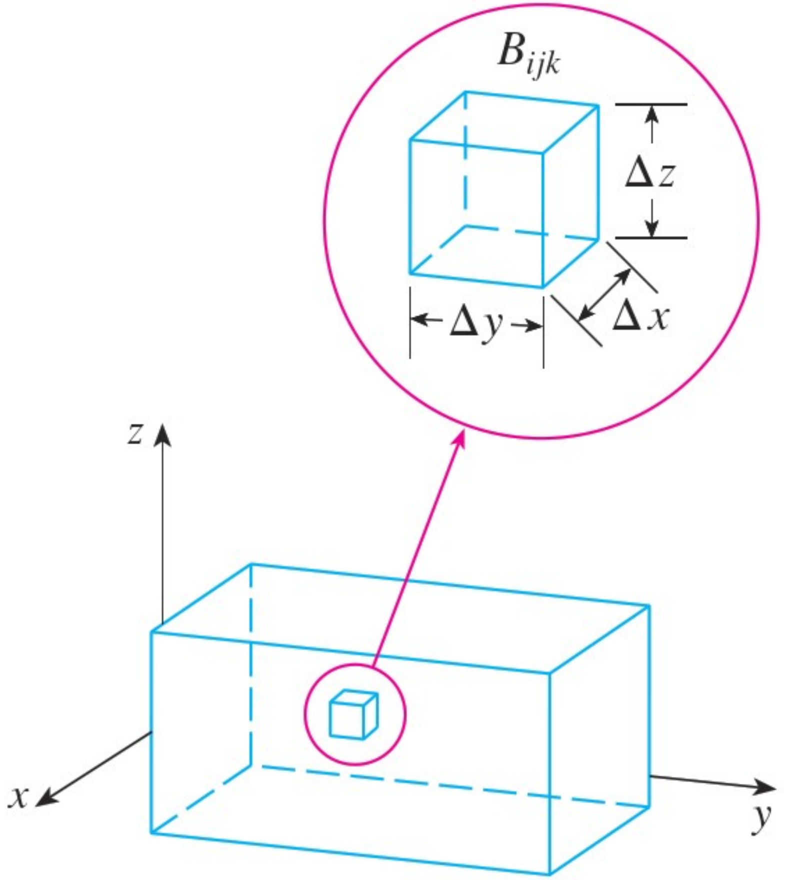


$$B = \{(x, y, z) \mid a \leqslant x \leqslant b,\ c \leqslant y \leqslant d,\ r \leqslant z \leqslant s\}$$

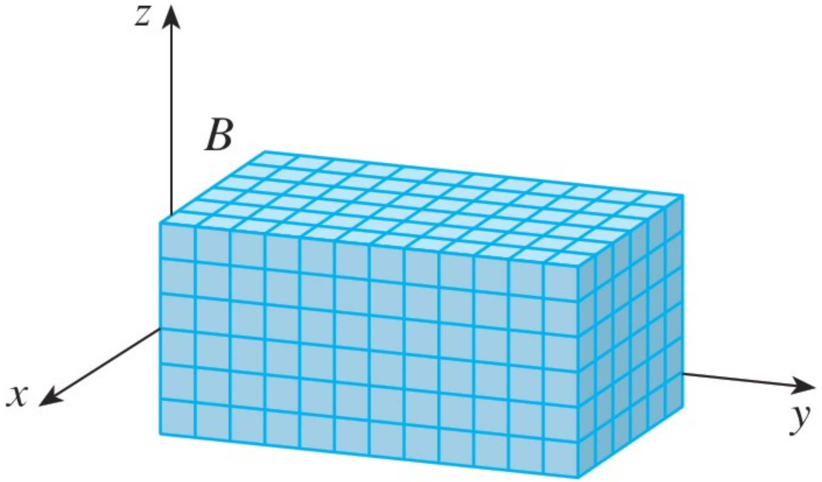
$$= [a,b] \times [c,d] \times [r,s]$$

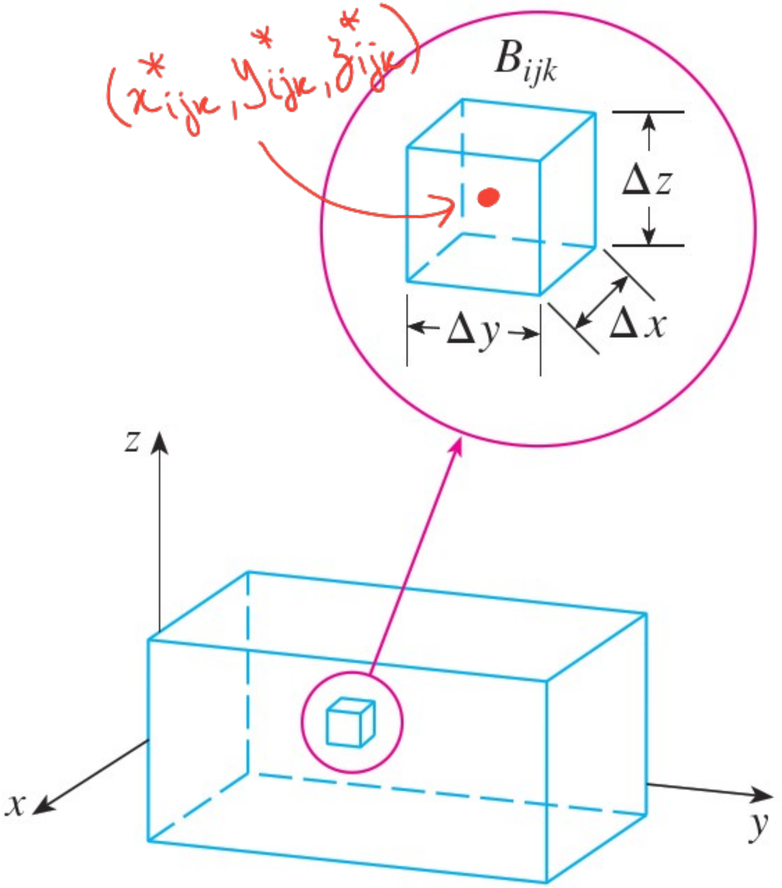
Integral tripla





Integral tripla





$$\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})\,\underbrace{\Delta x\,\Delta y\,\Delta z}_{\Delta V}$$

Integral tripla

A **integral tripla** de *f* na caixa *B* é

$$\iiint_B f(x, y, z)\, dV = \lim_{l,\, m,\, n \to \infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\, \Delta V$$

se esse limite existir.

Teorema de Fubini

Se f é contínua em uma caixa retangular $B = [a, b] \times [c, d] \times [r, s]$, então

$$\iiint_B f(x, y, z)\, dV = \int_r^s \int_c^d \int_a^b f(x, y, z)\, dx\, dy\, dz$$

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Existem cinco outras ordens possíveis de integração

Exemplo

Calcule a integral tripla $\iiint_B xyz^2\,dV$, onde B é a caixa retangular dada por

$$B = \{(x, y, z) \mid 0 \leqslant x \leqslant 1,\ -1 \leqslant y \leqslant 2,\ 0 \leqslant z \leqslant 3\}$$

$$\iiint_B xyz^2\,dV = \int_{-1}^{2}\int_{0}^{3}\left[\int_{0}^{1} xyz^2\,dx\right]dz\,dy = \int_{-1}^{2}\int_{0}^{3} yz^2\left(\int_{0}^{1} x\,dx\right)dz\,dy$$

$$= \int_{0}^{1} x\,dx \cdot \int_{-1}^{2}\left(\int_{0}^{3} yz^2\,dz\right)dy = \int_{0}^{1} x\,dx \cdot \int_{0}^{3} z^2\,dz \cdot \int_{-1}^{2} y\,dy = \frac{x^2}{2}\Big|_0^1 \cdot \frac{z^3}{3}\Big|_0^3 \cdot \frac{y^2}{2}\Big|_{-1}^2$$

$$= \frac{1}{2}\cdot 9 \cdot \left(2 - \frac{1}{2}\right) = \frac{9}{2}\cdot\frac{3}{2} = \frac{27}{4}$$

Exemplo

Calcule a integral tripla $\iiint_B xyz^2\,dV$, onde B é a caixa retangular dada por

$$B = \left\{(x, y, z) \mid 0 \leqslant x \leqslant 1,\ -1 \leqslant y \leqslant 2,\ 0 \leqslant z \leqslant 3\right\}$$

$$\iiint_B xyz^2\,dV = \int_0^3 \int_{-1}^2 \int_0^1 xyz^2\,dx\,dy\,dz$$

Exemplo

Calcule a integral tripla $\iiint_B xyz^2\, dV$, onde B é a caixa retangular dada por

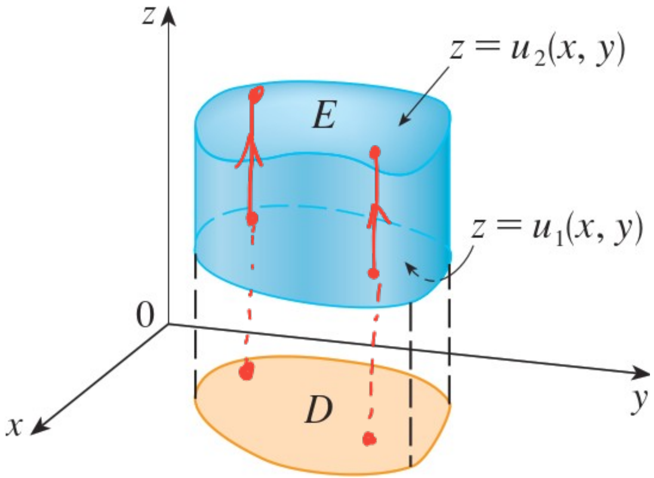
$$B = \{(x, y, z) \mid 0 \leqslant x \leqslant 1,\ -1 \leqslant y \leqslant 2,\ 0 \leqslant z \leqslant 3\}$$

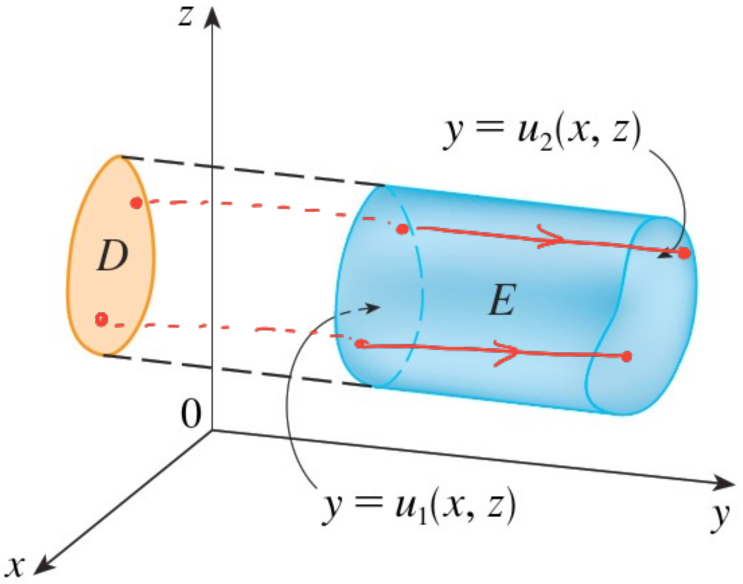
$$\iiint_B xyz^2\, dV = \int_0^3 \int_{-1}^2 \int_0^1 xyz^2\, dx\, dy\, dz = \int_0^3 \int_{-1}^2 \left[\frac{x^2yz^2}{2} \right]_{x=0}^{x=1} dy\, dz$$

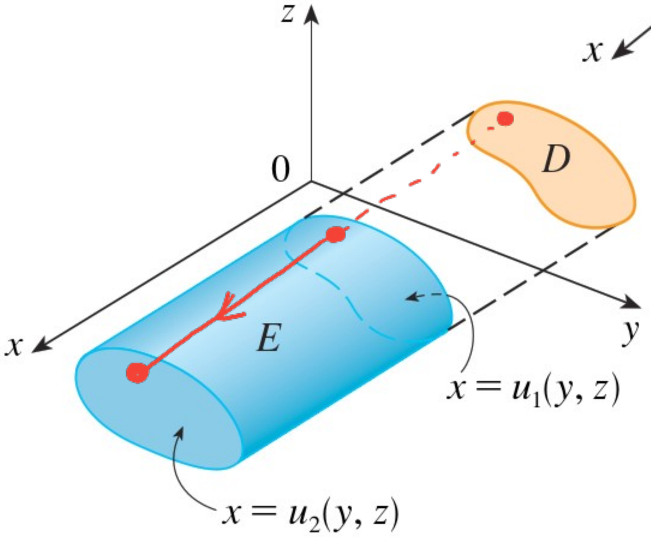
$$= \int_0^3 \int_{-1}^2 \frac{yz^2}{2}\, dy\, dz = \int_0^3 \left[\frac{y^2z^2}{4} \right]_{y=-1}^{y=2} dz$$

$$= \int_0^3 \frac{3z^2}{4}\, dz = \left. \frac{z^3}{4} \right]_0^3 = \frac{27}{4}$$

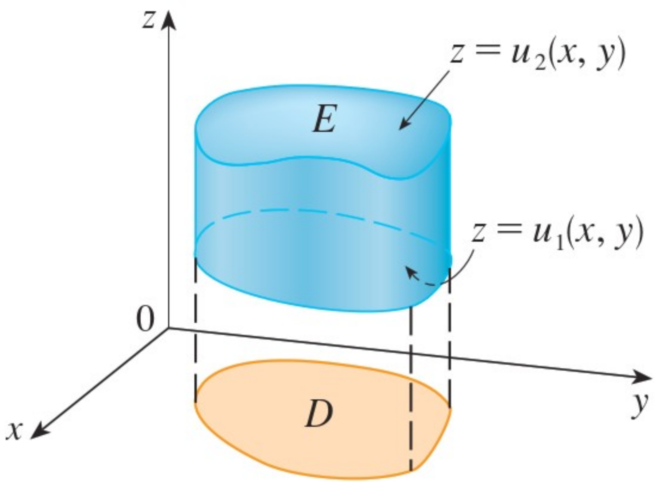
Regiões gerais





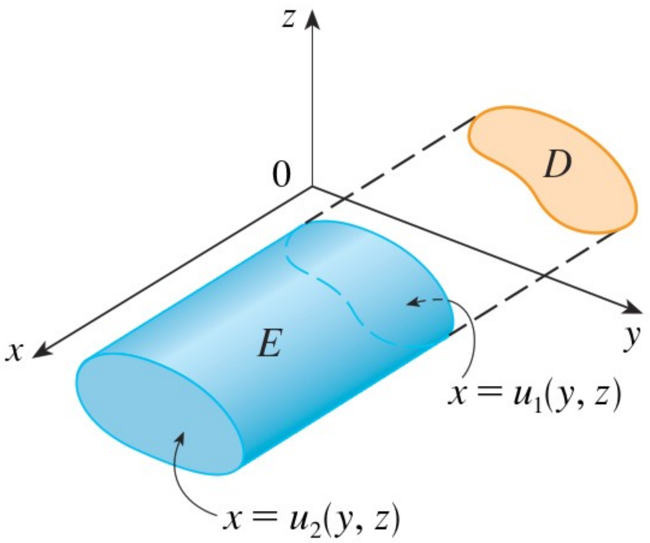


Regiões gerais: tipo I



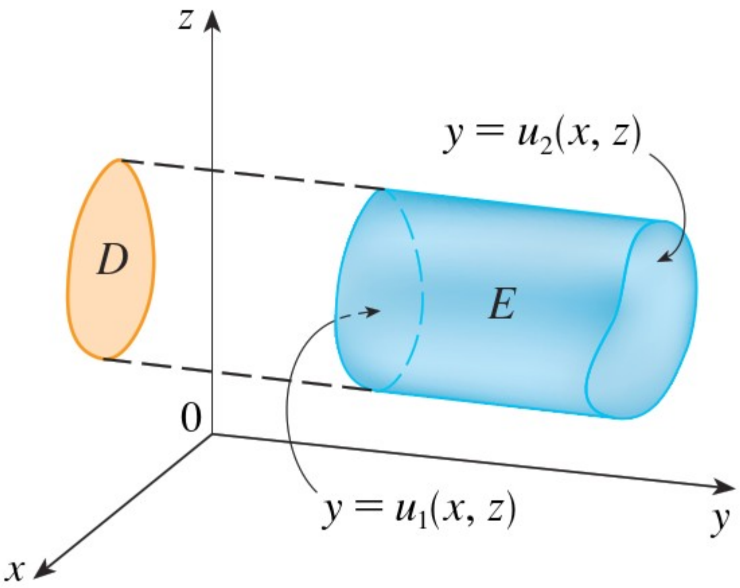
$$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z)\, dz \right] dA$$

Regiões gerais: tipo II



$$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x, y, z)\, dx \right] dA$$

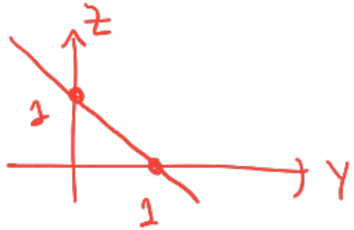
Regiões gerais: tipo III



$$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x, y, z)\, dy \right] dA$$

Exemplo

Calcule $\iiint_E z \, dV$, onde E é o tetraedro sólido limitado pelos quatro planos $x = 0, y = 0, z = 0$ e $x + y + z = 1$.



(yz) (xz) (xy)

Inters. $x=0$ e $x+y+z=1$: $y+z=1 \Leftrightarrow z=-y+1$

" $y=0$ e " : $x+z=1 \Leftrightarrow z=-x+1$

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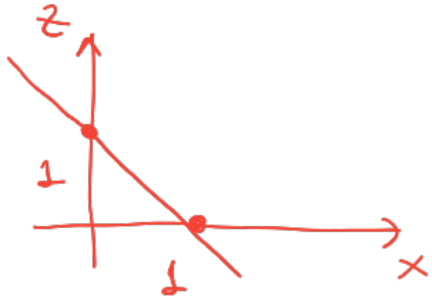
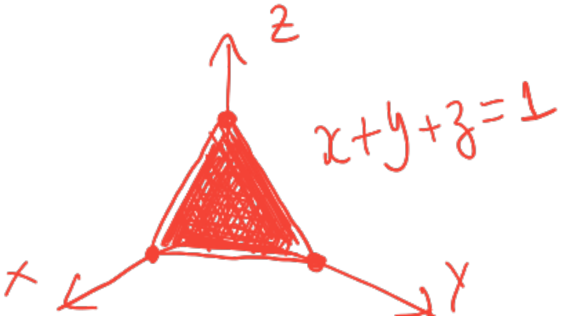
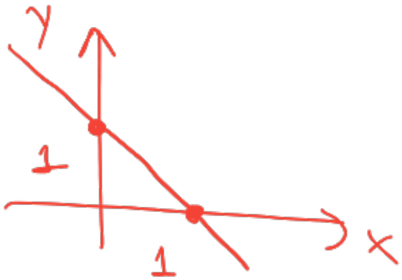


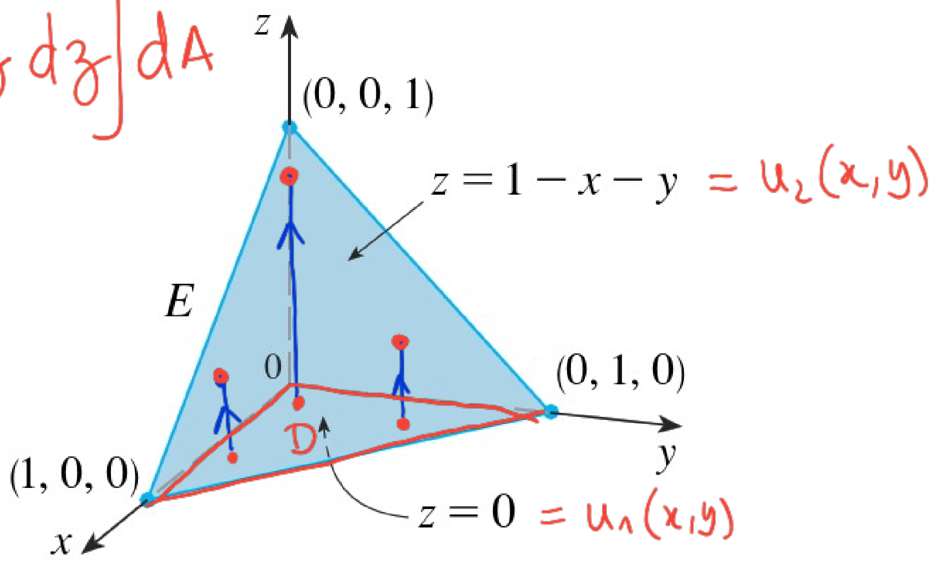




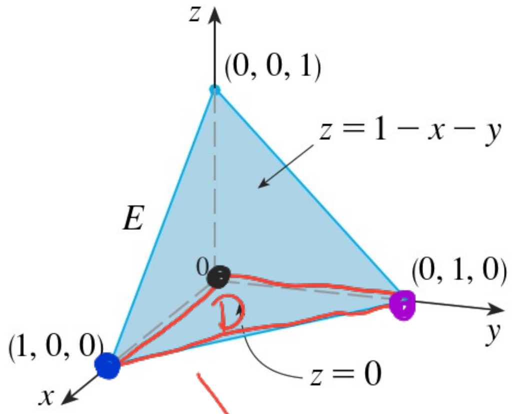
Exemplo

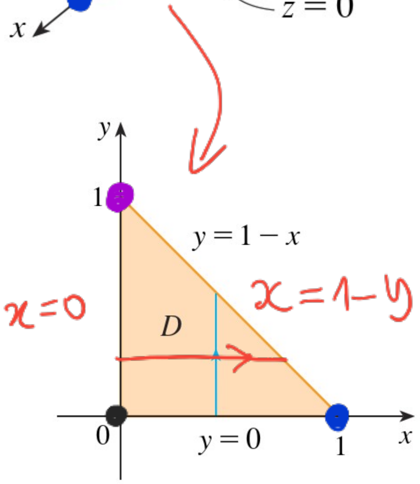
Calcule $\iiint_E z\,dV$, onde E é o tetraedro sólido limitado pelos quatro planos $x = 0$, $y = 0$, $z = 0$ e $x + y + z = 1$.

$$\iiint_E z\,dV = \iint_D \left[\int_0^{1-x-y} z\,dz \right] dA$$



Exemplo





$$\iiint_E z\,dV = \iint_D \left[\int_0^{1-x-y} z\,dz\right] dA \quad (E \text{ tipo I})$$

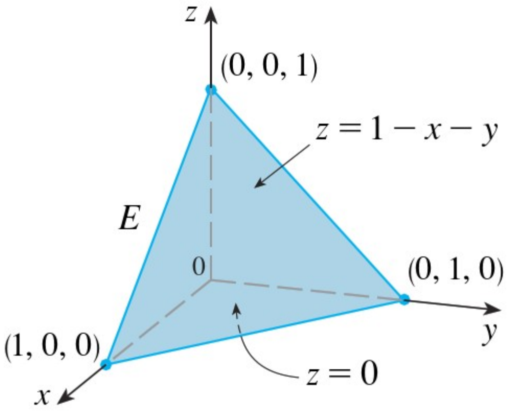
$$= \int_0^1 \int_0^{1-x} \left[\int_0^{1-x-y} z\,dz\right] dy\,dx \quad (D \text{ tipo I})$$

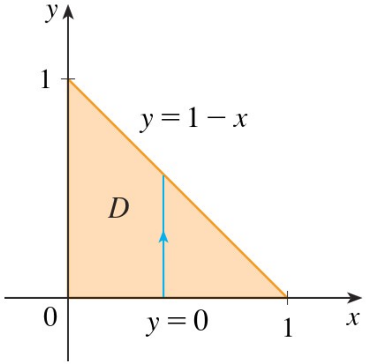
$$= \int_0^1 \int_0^{1-y} \left[\int_0^{1-x-y} z\,dz\right] dx\,dy \quad (D \text{ tipo II})$$

$$E = \{(x,y,z) \mid 0 \le x \le 1,\ 0 \le y \le 1-x,\ 0 \le z \le 1-x-y\}$$

$$= \{(x,y,z) \mid 0 \le y \le 1,\ 0 \le x \le 1-y,\ 0 \le z \le 1-x-y\}$$

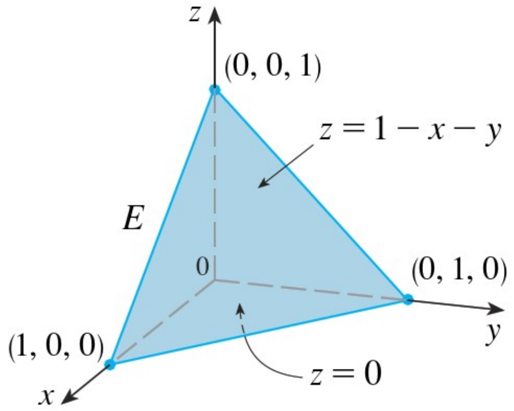
Exemplo

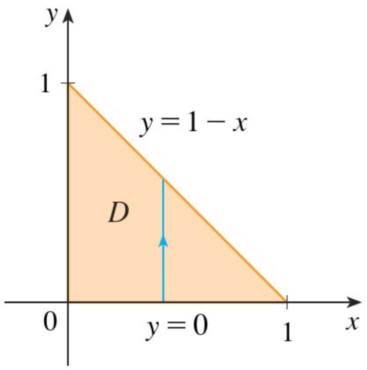




$$E = \{(x, y, z) \mid 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1 - x,\ 0 \leqslant z \leqslant 1 - x - y\}$$

Exemplo

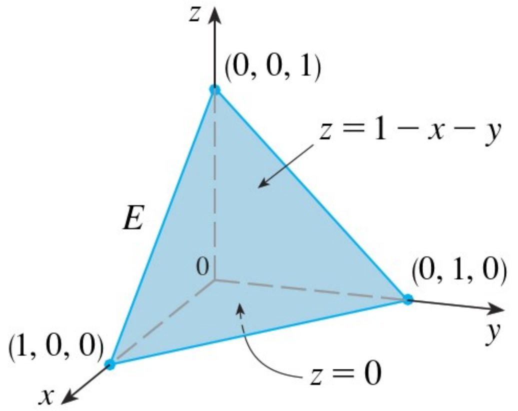


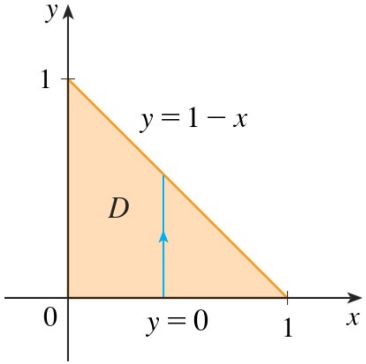


$$E = \{(x, y, z) \mid 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1 - x,\ 0 \leqslant z \leqslant 1 - x - y\}$$

$$\iiint_E z\, dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} z\, dz\, dy\, dx$$

Exemplo





$$\int (1-x-y)^2\,dy = \int u^2(-1)\,du = -\int u^2\,du = -\frac{u^3}{3} + C$$

$$= -\frac{(1-x-y)^3}{3} + C$$

$u = 1-x-y$

$du = (-1)\,dy$

$\Rightarrow dy = -du$

$$E = \{(x, y, z) \mid 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1 - x,\ 0 \leqslant z \leqslant 1 - x - y\}$$

$$\iiint_E z\,dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} z\,dz\,dy\,dx = \int_0^1 \int_0^{1-x} \left[\frac{z^2}{2}\right]_{z=0}^{z=1-x-y} dy\,dx$$

$$= \tfrac{1}{2} \int_0^1 \int_0^{1-x} (1 - x - y)^2\,dy\,dx = \tfrac{1}{2} \int_0^1 \left[-\frac{(1 - x - y)^3}{3}\right]_{y=0}^{y=1-x} dx$$

$$= \tfrac{1}{6} \int_0^1 (1 - x)^3\,dx = \frac{1}{6}\left[-\frac{(1 - x)^4}{4}\right]_0^1 = \frac{1}{24}$$

Exemplo

Calcule $\iiint_E \sqrt{x^2 + z^2}\, dV$, onde E é a região limitada pelo paraboloide $y = x^2 + z^2$ e pelo plano $y = 4$.

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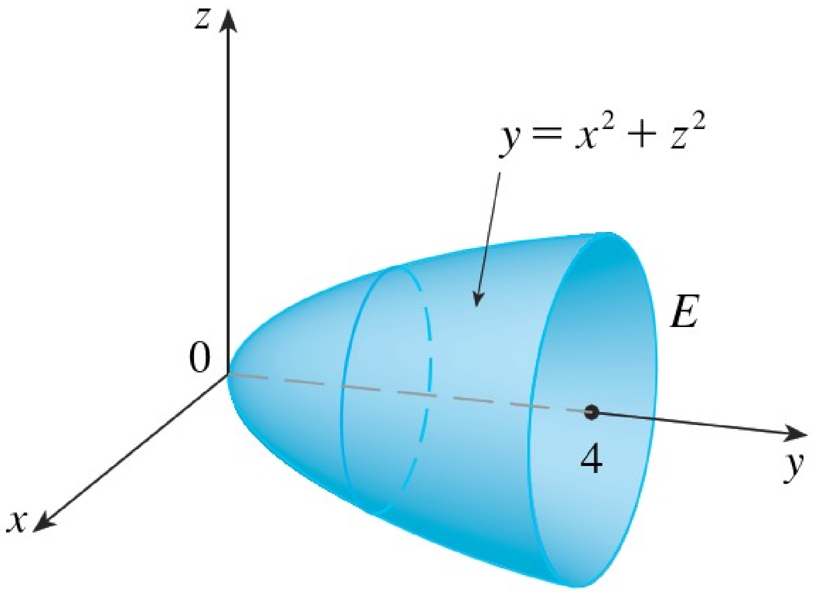
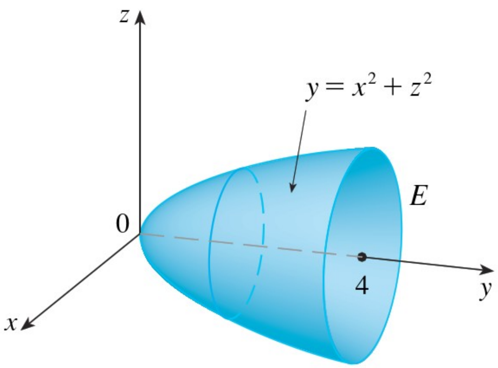
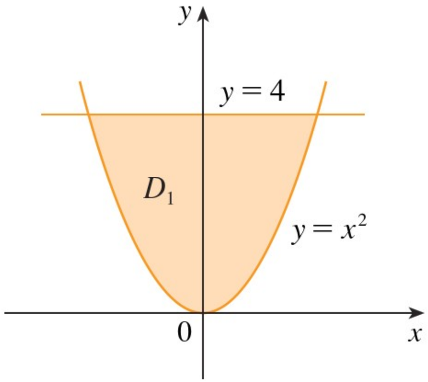


Exemplo (tipo I)





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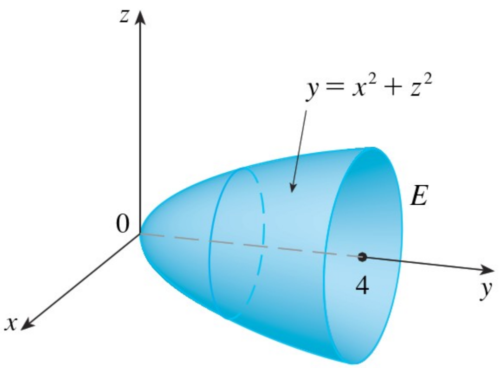
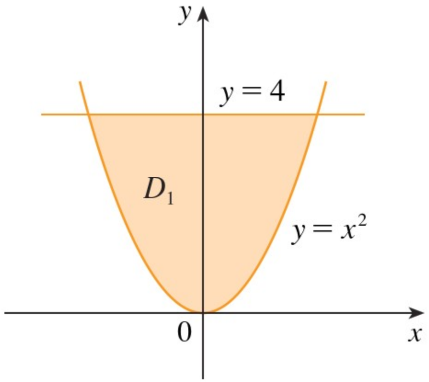
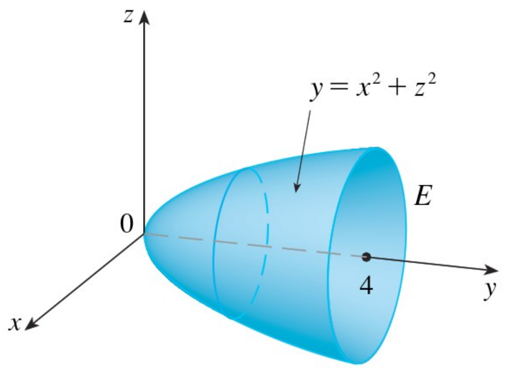


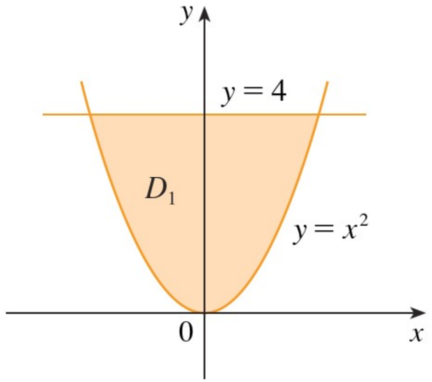


De $y = x^2 + z^2$ obtemos $z = \pm\sqrt{y - x^2}$, e então a superfície limite de baixo de E é $z = -\sqrt{y - x^2}$ e a superfície de cima é $z = \sqrt{y - x^2}$. Portanto, a descrição de E como região do tipo 1 é

$$E = \left\{(x, y, z) \mid -2 \leqslant x \leqslant 2,\ x^2 \leqslant y \leqslant 4,\ -\sqrt{y - x^2} \leqslant z \leqslant \sqrt{y - x^2}\right\}$$

Exemplo (tipo I)





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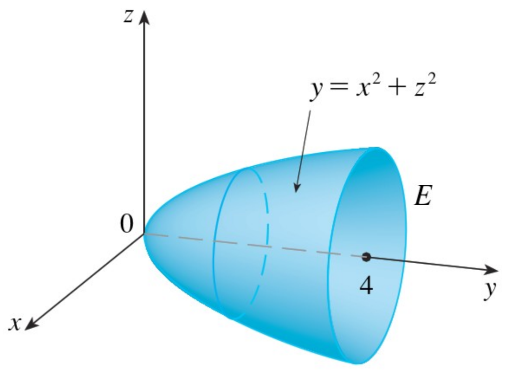
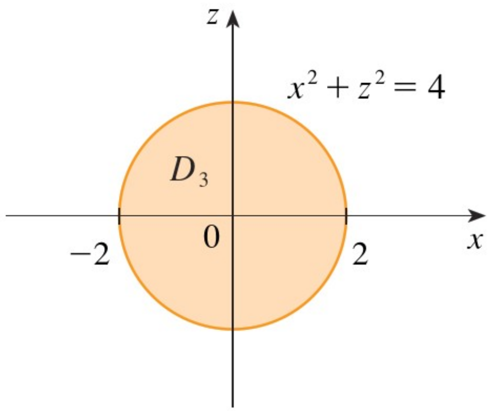
$$E = \left\{(x, y, z) \mid -2 \leqslant x \leqslant 2,\ x^2 \leqslant y \leqslant 4,\ -\sqrt{y - x^2} \leqslant z \leqslant \sqrt{y - x^2}\right\}$$

e obtemos

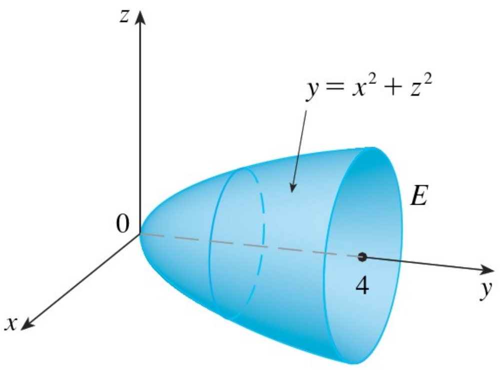
$$\iiint_E \sqrt{x^2 + z^2}\, dV = \int_{-2}^{2} \int_{x^2}^{4} \int_{-\sqrt{y - x^2}}^{\sqrt{y - x^2}} \sqrt{x^2 + z^2}\, dz\, dy\, dx$$

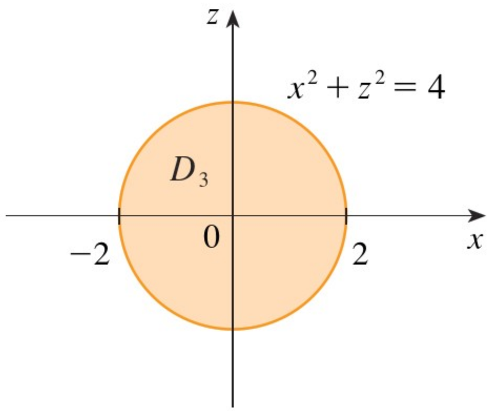
Exemplo (tipo III)





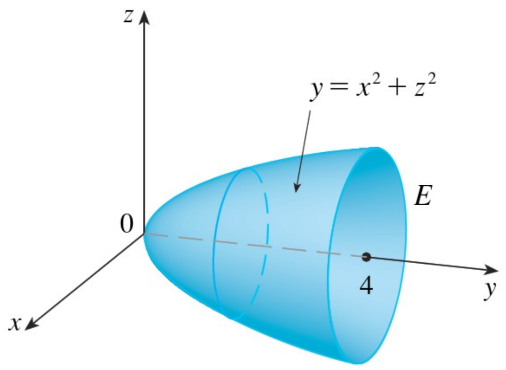
Exemplo (tipo III)

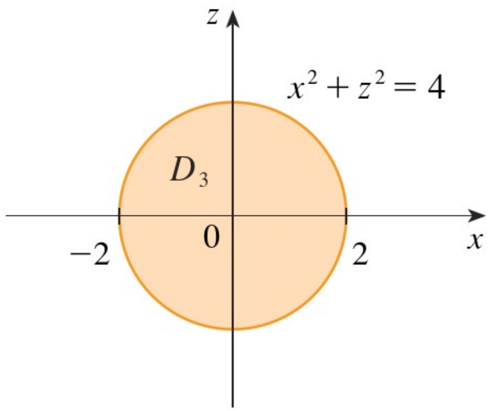




$$\iiint_E \sqrt{x^2 + z^2}\, dV = \iint_{D_3} \left[\int_{x^2+z^2}^{4} \sqrt{x^2 + z^2}\, dy \right]$$

Exemplo (tipo III)





$$\iiint_E \sqrt{x^2 + z^2}\, dV = \iint_{D_3} \left[\int_{x^2+z^2}^{4} \sqrt{x^2 + z^2}\, dy \right] = \iint_{D_3} (4 - x^2 - z^2)\sqrt{x^2 + z^2}\, dA$$

$$= \int_0^{2\pi} \int_0^2 (4 - r^2) r\, r\, dr\, d\theta = \int_0^{2\pi} d\theta \int_0^2 (4r^2 - r^4)\, dr$$

$$= 2\pi \left[\frac{4r^3}{3} - \frac{r^5}{5} \right]_0^2 = \frac{128\pi}{15}$$

Exercício

Calcule $\iiint_E z\, dV$, onde E é o tetraedro sólido limitado pelos quatro planos $x = 0$, $y = 0$, $z = 0$ e $x + y + z = 1$.