Cálculo III

Integral sobre regiões gerais

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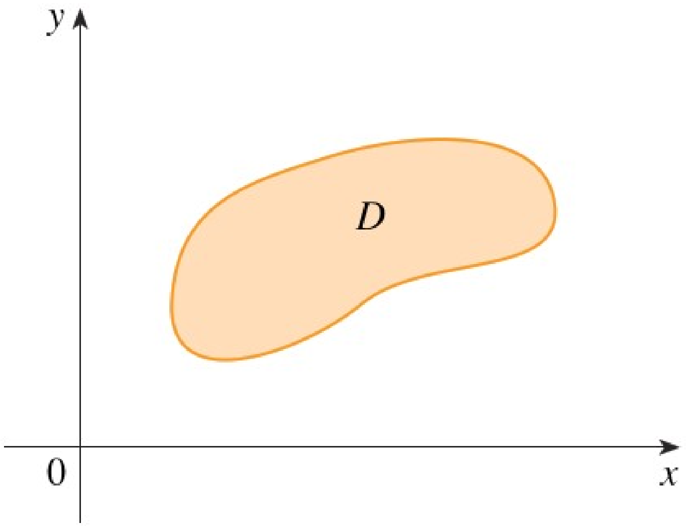
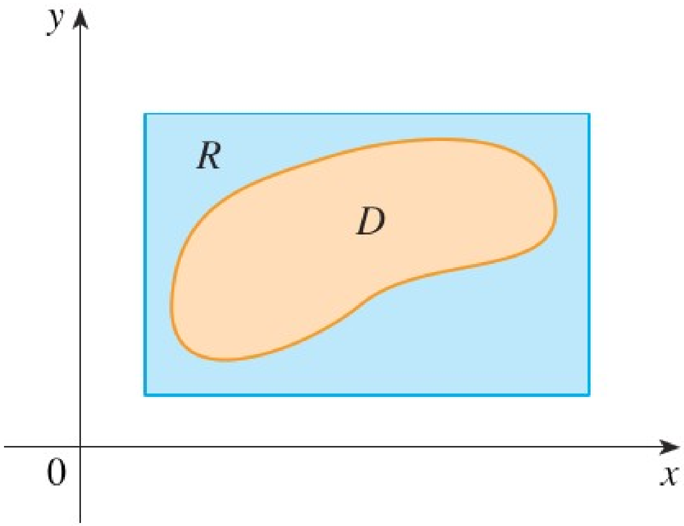
Integrais sobre regiões gerais

$$F(x, y) = \begin{cases} f(x, y) & \text{se } (x, y) \text{ está em } D \\ 0 & \text{se } (x, y) \text{ está em } R \text{ mas não em } D \end{cases}$$





Integrais sobre regiões gerais

Se F for integrável em R, então definimos a **integral dupla de f em D** por

$$\iint_D f(x, y)\, dA = \iint_R F(x, y)\, dA$$

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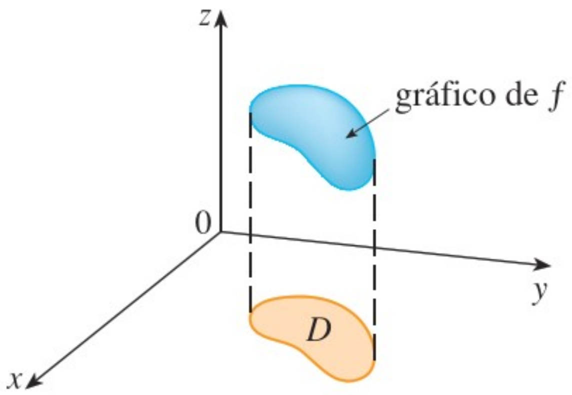
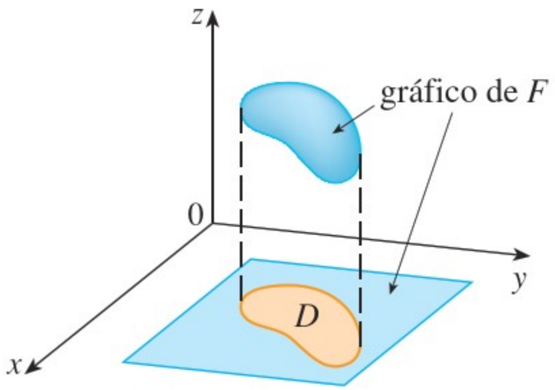
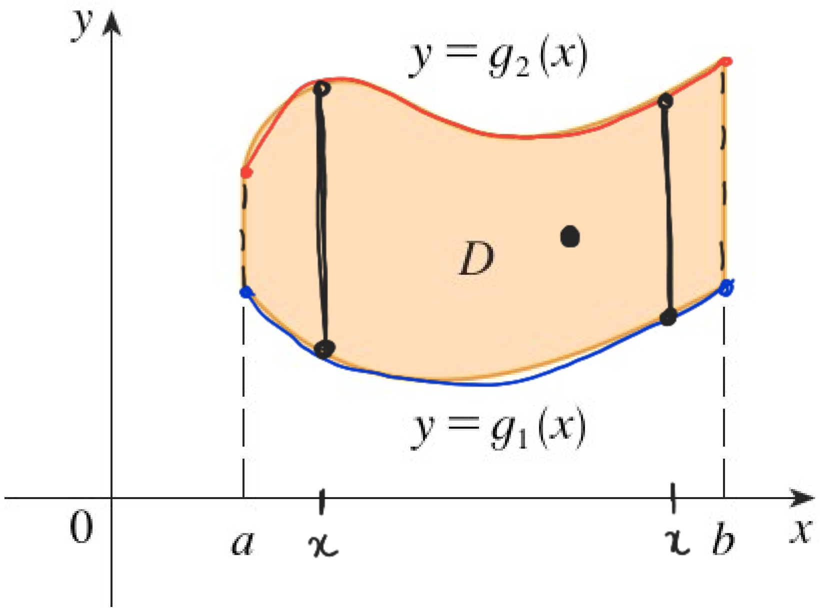
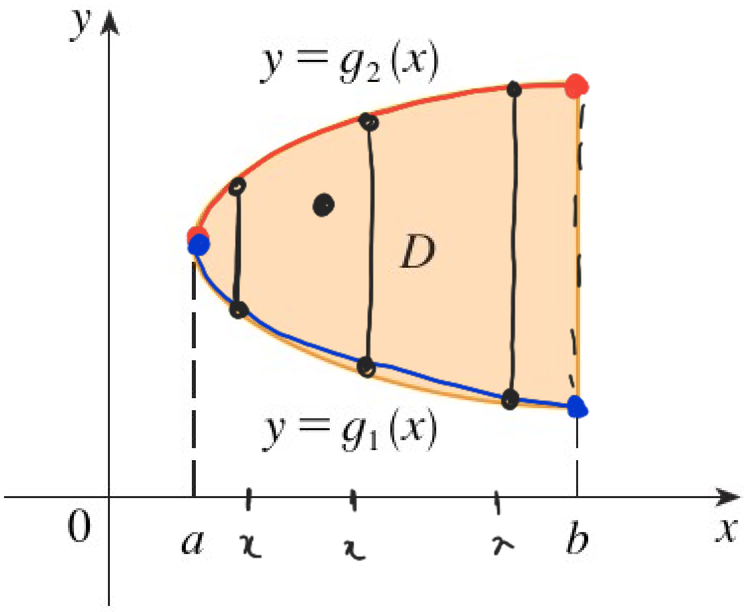




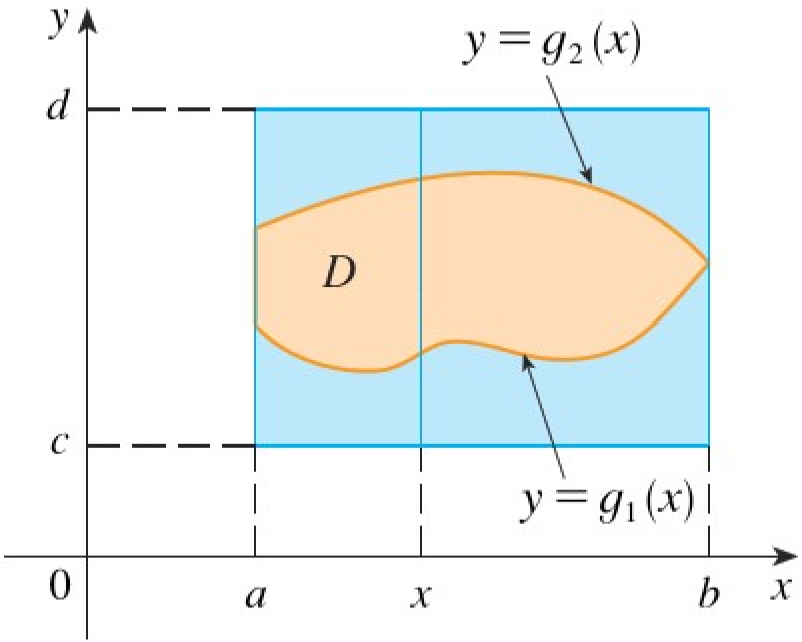
Regiões Tipo I

$$D = \{(x, y) \mid a \leqslant x \leqslant b,\ g_1(x) \leqslant y \leqslant g_2(x)\}$$



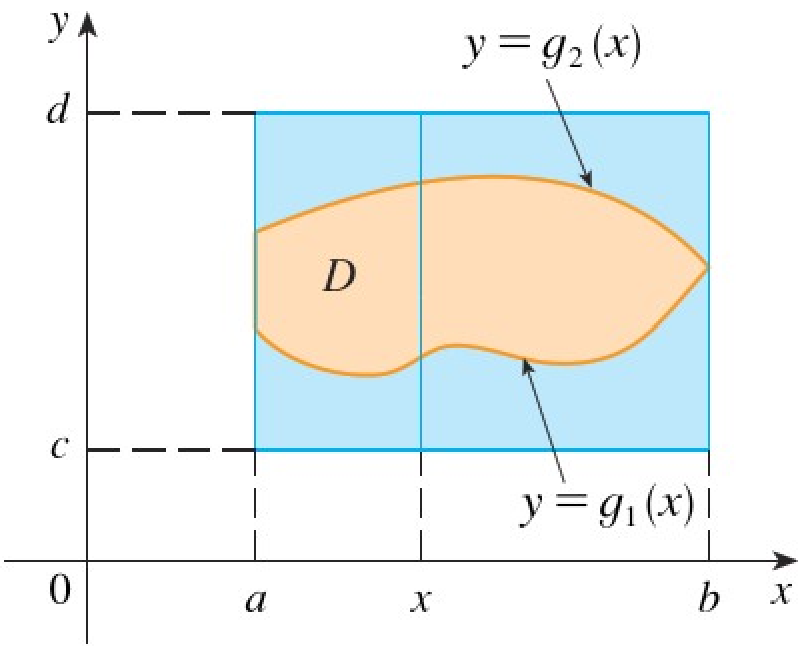


Regiões Tipo I



$$\iint_D f(x, y)\, dA = \iint_R F(x, y)\, dA = \int_a^b \int_c^d F(x, y)\, dy\, dx$$

Regiões Tipo I

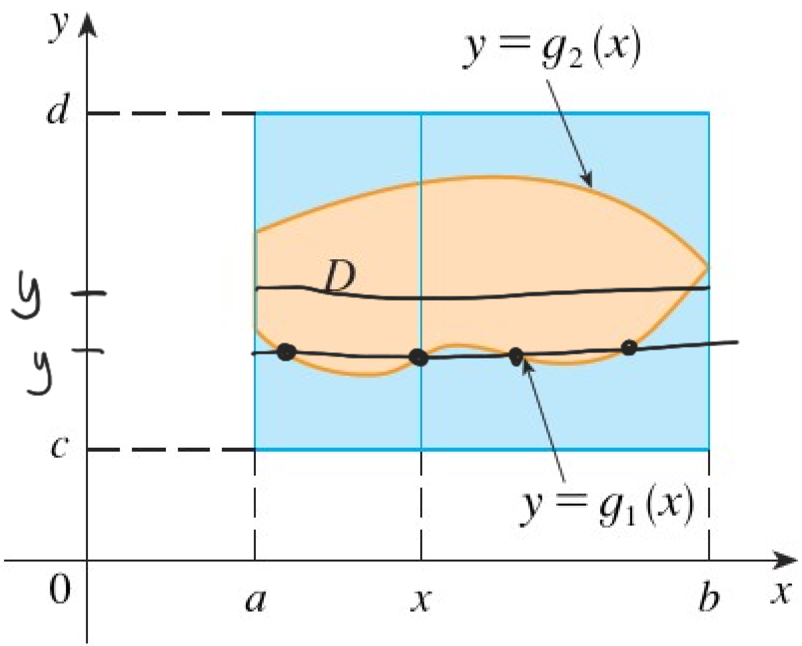


$$\iint_D f(x, y)\, dA = \iint_R F(x, y)\, dA = \int_a^b \int_c^d F(x, y)\, dy\, dx$$

Observe que $F(x, y) = 0$ se $y < g_1(x)$ ou $y > g_2(x)$

$$\int_c^d F(x, y)\, dy = \int_{g_1(x)}^{g_2(x)} F(x, y)\, dy = \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy$$

Regiões Tipo I



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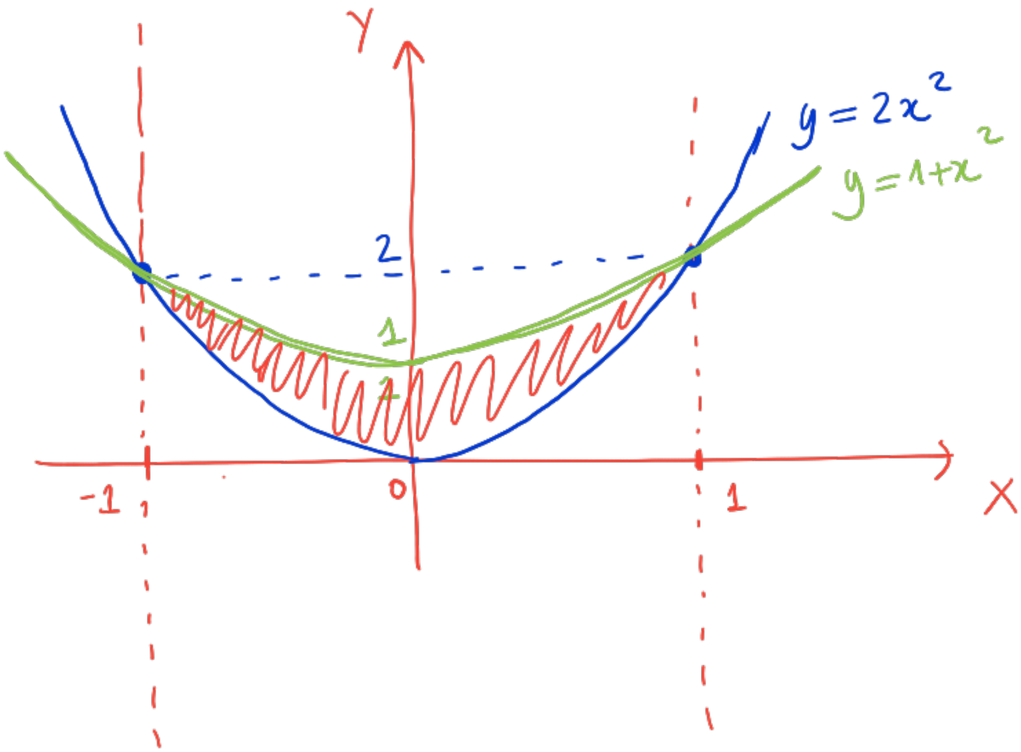
Se f é contínua em uma região D do tipo I tal que
$D = \{(x, y) \mid a \leqslant x \leqslant b,\ g_1(x) \leqslant y \leqslant g_2(x)\}$ então,

$$\iint_D f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx$$

Exemplo:

Calcule $\iint_D (x + 2y)\, dA$, onde $D = \{(x, y) \mid -1 \leqslant x \leqslant 1,\ 2x^2 \leqslant y \leqslant 1 + x^2\}$

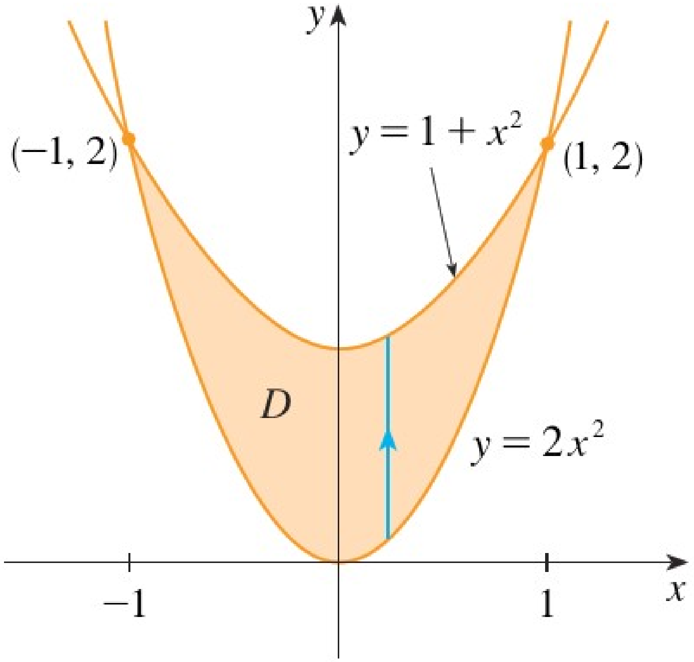


$$2x^2 = 1 + x^2$$

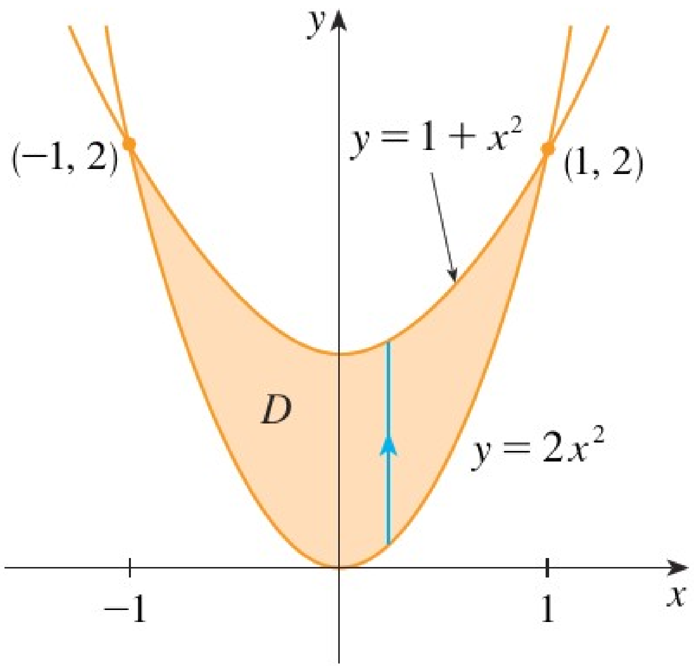
$$\Rightarrow x^2 = 1 \Rightarrow x = \pm 1$$

Exemplo:

Calcule $\iint_D (x + 2y)\, dA$, onde $D = \left\{(x, y) \mid -1 \leqslant x \leqslant 1,\ 2x^2 \leqslant y \leqslant 1 + x^2\right\}$

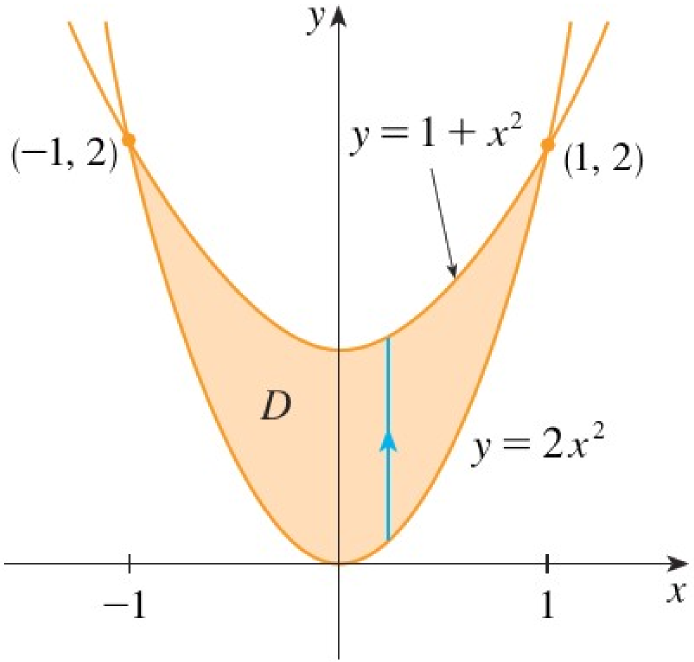


Exemplo:



$$\iint_D (x + 2y)\, dA = \int_{-1}^{1} \int_{2x^2}^{1+x^2} (x + 2y)\, dy\, dx$$

Exemplo:



$$\iint_D (x+2y)\,dA = \int_{-1}^{1}\int_{2x^2}^{1+x^2} (x+2y)\,dy\,dx$$

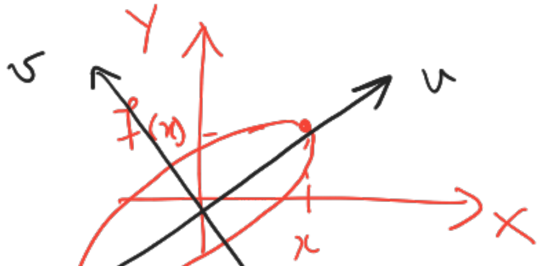
$$= \int_{-1}^{1} \left[xy + y^2\right]_{y=2x^2}^{y=1+x^2} dx$$

$$= \int_{-1}^{1} \left[x(1+x^2) + (1+x^2)^2 - x(2x^2) - (2x^2)^2\right] dx$$

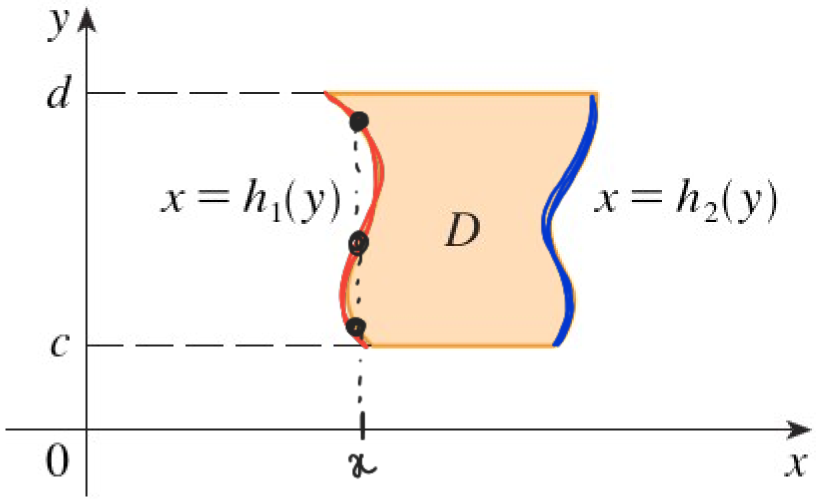
$$= \int_{-1}^{1} (-3x^4 - x^3 + 2x^2 + x + 1)\,dx$$

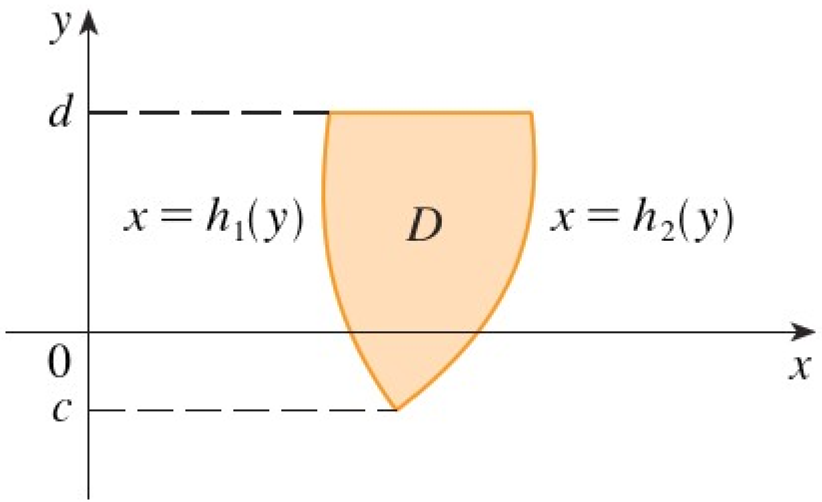
$$= -3\,\frac{x^5}{5} - \frac{x^4}{4} + 2\,\frac{x^3}{3} + \frac{x^2}{2} + x\Bigg]_{-1}^{1} = \frac{32}{15}$$

Regiões Tipo II



$$D = \{(x, y) \mid c \leqslant y \leqslant d,\ h_1(y) \leqslant x \leqslant h_2(y)\}$$





Regiões Tipo II

$$D = \left\{(x, y) \mid c \leqslant y \leqslant d,\ h_1(y) \leqslant x \leqslant h_2(y)\right\}$$

$$\iint_D f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy$$

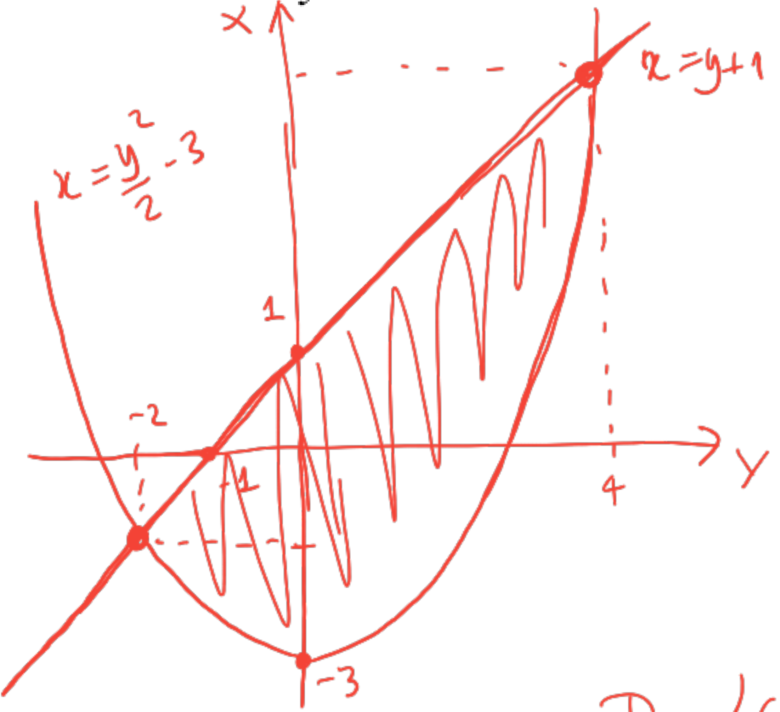
onde D é uma região do tipo II

Exemplo:

$x = y + 1$

Calcule $\iint_D xy \, dA$, onde D é a região limitada pela reta $y = x - 1$ pela parábola $y^2 = 2x + 6$.

$$x = \frac{y^2 - 6}{2} = \frac{y^2}{2} - 3$$

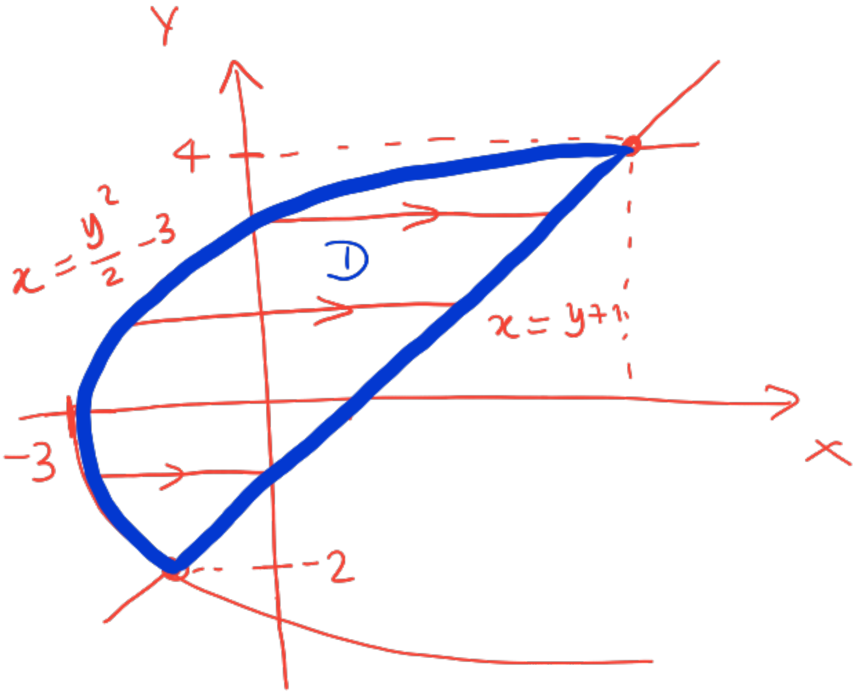


$$y + 1 = \frac{y^2}{2} - 3$$

$$2y + 2 = y^2 - 6$$

$$y^2 - 2y - 8 = 0$$

$$y = 4 \text{ ou } y = -2$$

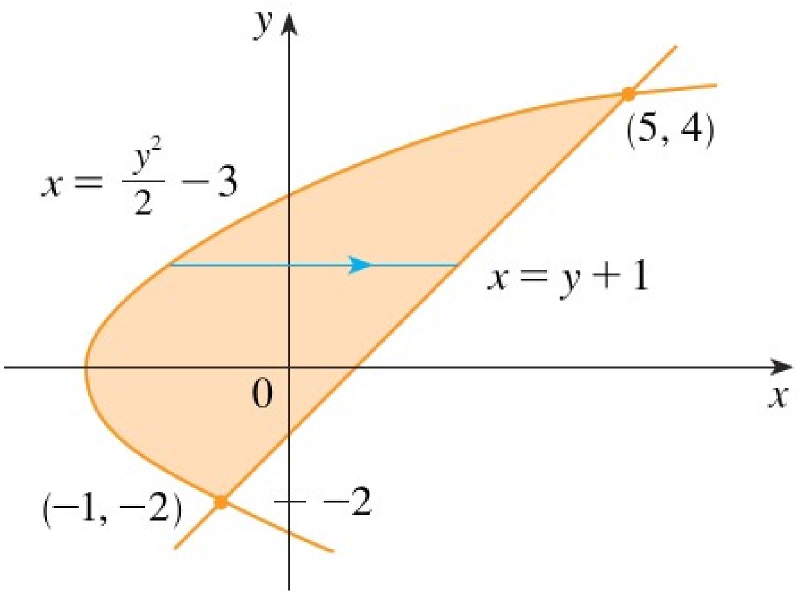


$$D = \left\{ (x, y) \mid -2 \le y \le 4, \ \frac{y^2}{2} - 3 \le x \le y + 1 \right\}$$

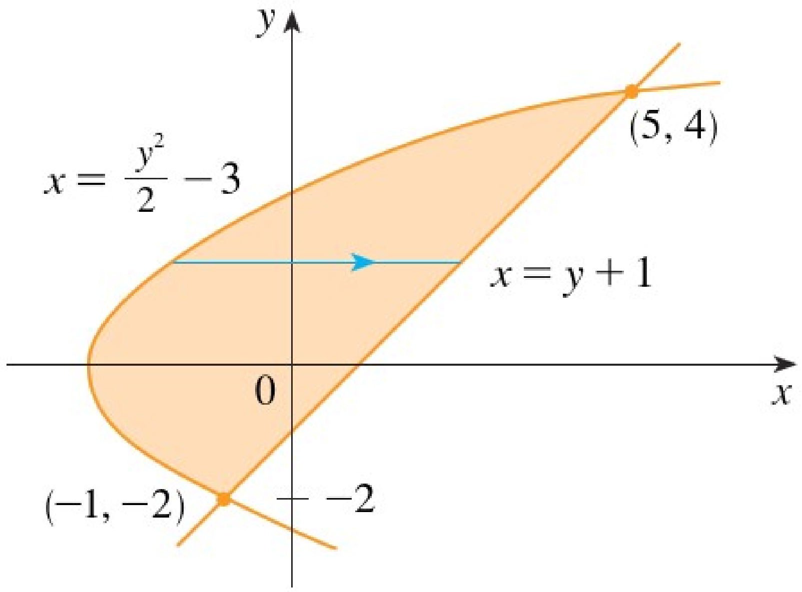
Exemplo:

Calcule $\iint_D xy\, dA$, onde D é a região limitada pela reta $y = x - 1$ pela parábola $y^2 = 2x + 6$.

$$D = \left\{(x, y) \mid -2 \leqslant y \leqslant 4,\ \tfrac{1}{2}y^2 - 3 \leqslant x \leqslant y + 1\right\}$$

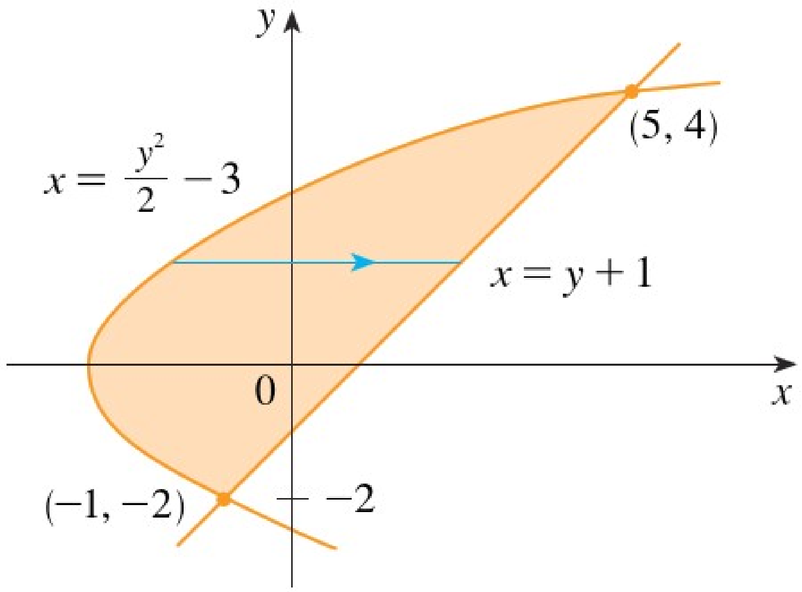


Exemplo:



$$\iint_D xy\, dA = \int_{-2}^{4} \int_{\frac{1}{2}y^2-3}^{y+1} xy\, dx\, dy$$

Exemplo:



$$\iint_D xy\, dA = \int_{-2}^{4} \int_{\frac{1}{2}y^2-3}^{y+1} xy\, dx\, dy$$

$$= \int_{-2}^{4} \left[\frac{x^2}{2} y \right]_{x=\frac{1}{2}y^2-3}^{x=y+1} dy$$

$$= \frac{1}{2} \int_{-2}^{4} y\left[(y+1)^2 - \left(\tfrac{1}{2}y^2 - 3\right)^2\right] dy$$

$$= \frac{1}{2} \int_{-2}^{4} \left(-\frac{y^5}{4} + 4y^3 + 2y^2 - 8y \right) dy$$

$$= \frac{1}{2} \left[-\frac{y^6}{24} + y^4 + 2\frac{y^3}{3} - 4y^2 \right]_{-2}^{4} = 36$$

Exercícios:

1. Calcule a integral dupla $\int\int_D y^2\, dA$, onde

$$D = \{(x, y) \mid -1 \leq y \leq 1, -y - 2 \leq x \leq y\}.$$

   Resp: $\frac{4}{3}$

2. Calcule a integral dupla $\int\int_D x\, dA$, onde

$$D = \{(x, y) \mid 0 \leq x \leq \pi, 0 \leq y \leq \text{sen} x\}.$$

   Resp: π