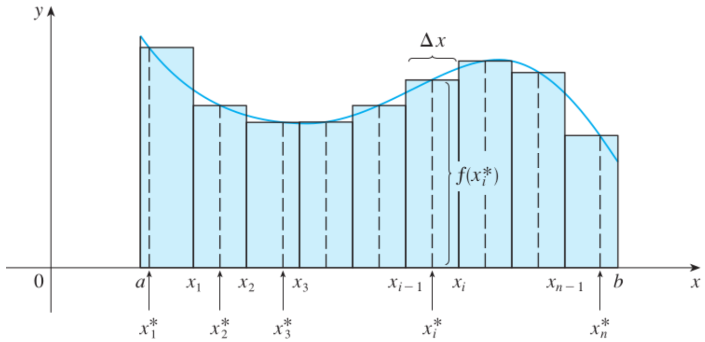
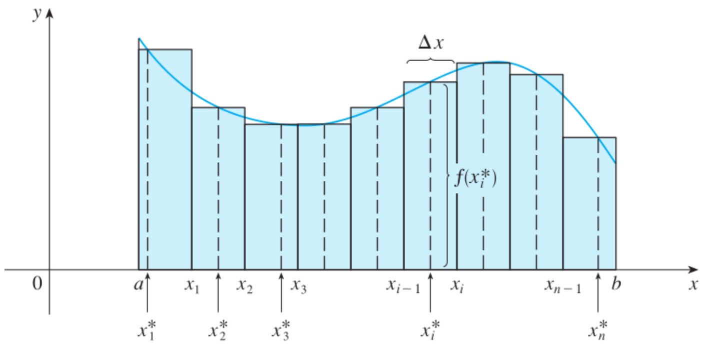
Cálculo III

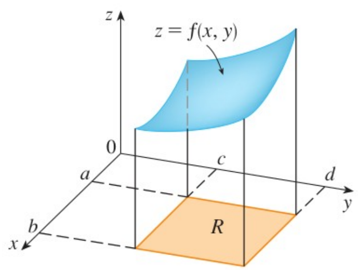
Integral dupla



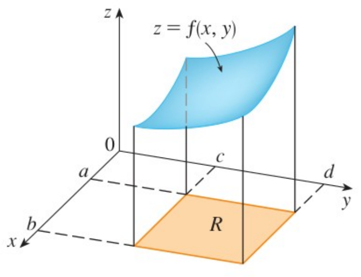


$$\sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$



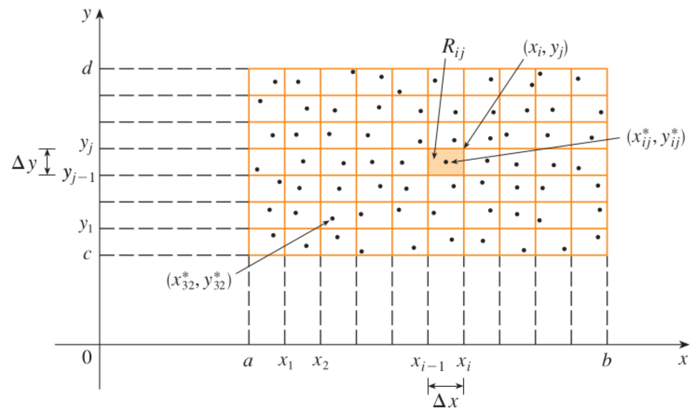
$$f(x, y) \geqslant 0$$



$$f(x, y) \geq 0$$

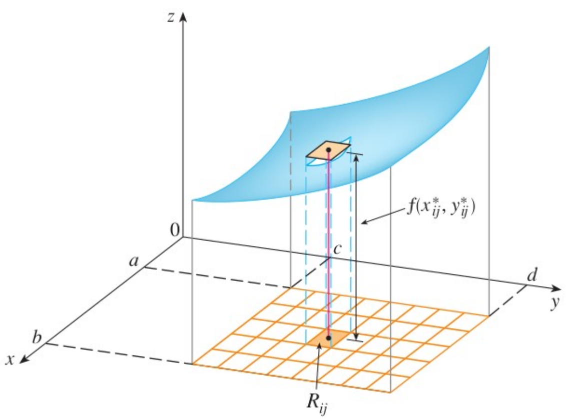
$$R = [a, b] \times [c, d] = \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b,\ c \leq y \leq d\}$$

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid 0 \leq z \leq f(x, y),\ (x, y) \in R\}$$



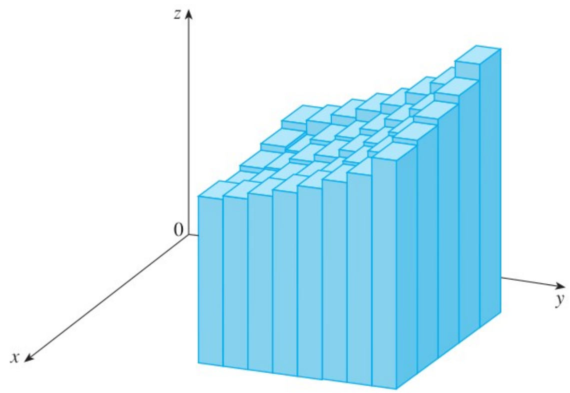
$$\Delta x = (b - a)/m \qquad \Delta y = (d - c)/n$$

$$R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] = \{(x, y) \mid x_{i-1} \leq x \leq x_i,\ y_{j-1} \leq y \leq y_j\}$$



$$\Delta A = \Delta x\, \Delta y$$

$$f(x_{ij}^*, y_{ij}^*)\, \Delta A$$



$$\sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\, \Delta A$$

$$V = \lim_{m, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\, \Delta A$$

Integral dupla:

$$\iint_R f(x, y)\, dA = \lim_{m, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\, \Delta A$$

Se $f(x, y) \geqslant 0$, o volume do sólido acima da região *R* e abaixo gráfico da função é dado por:

$$V = \iint_R f(x, y)\, dA$$

Exemplo: Estime a área do sólido acima da região *R* e abaixo do gráfico da função abaixo

$R = [0, 2] \times [0, 2]$ $f(x, y) = 16 - x^2 - 2y^2$

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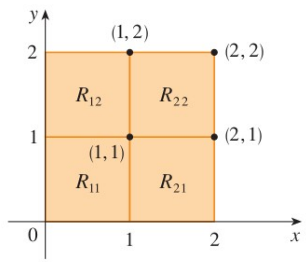

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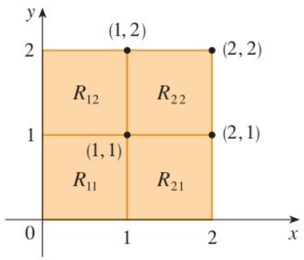
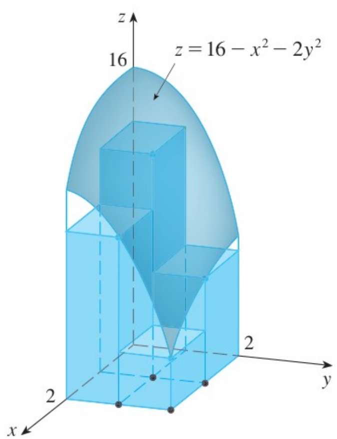


$$V \approx \sum_{i=1}^{2} \sum_{j=1}^{2} f(x_i, y_j)\, \Delta A$$



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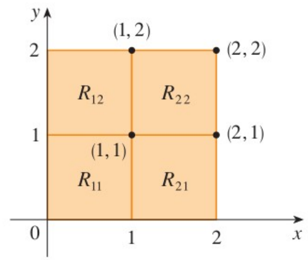
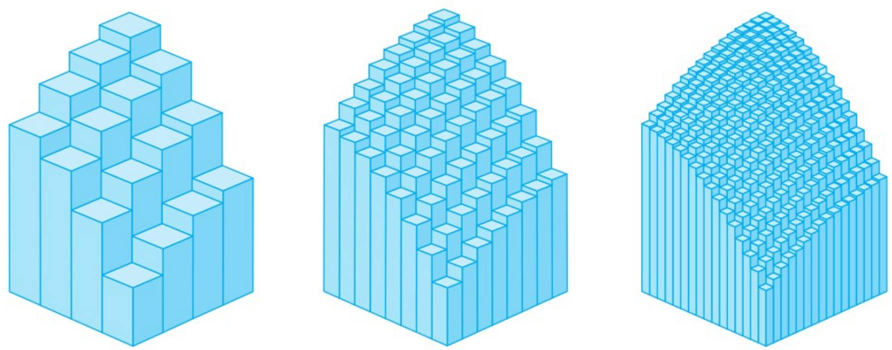


$$V \approx \sum_{i=1}^{2} \sum_{j=1}^{2} f(x_i, y_j)\, \Delta A = f(1, 1)\, \Delta A + f(1, 2)\, \Delta A + f(2, 1)\, \Delta A + f(2, 2)\, \Delta A$$

$$= 13(1) + 7(1) + 10(1) + 4(1) = 34$$



(a) $m = n = 4$, $V \approx 41.5$ (b) $m = n = 8$, $V \approx 44.875$ (c) $m = n = 16$, $V \approx 46.46875$

Exemplo: $R = \{(x, y) \mid -1 \leqslant x \leqslant 1, -2 \leqslant y \leqslant 2\}$

$$\iint_R \sqrt{1 - x^2}\, dA$$

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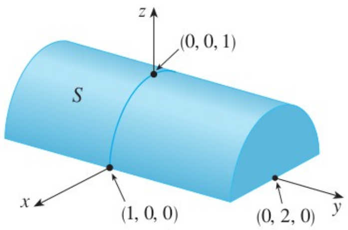

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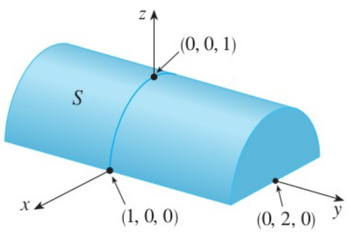


$$\iint_R \sqrt{1 - x^2}\, dA = \tfrac{1}{2}\pi(1)^2 \times 4 = 2\pi$$

Propriedades

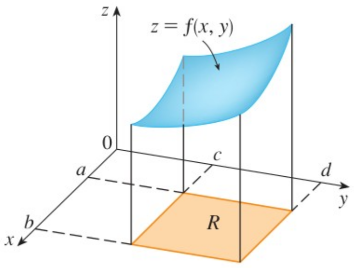
$$\iint_R [f(x, y) + g(x, y)]\, dA = \iint_R f(x, y)\, dA + \iint_R g(x, y)\, dA$$

$$\iint_R c\, f(x, y)\, dA = c \iint_R f(x, y)\, dA$$

$$f(x, y) \geqslant g(x, y)$$

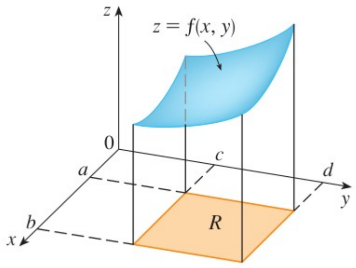
$$\iint_R f(x, y)\, dA \geqslant \iint_R g(x, y)\, dA$$

Integral Iterada



$R = [a, b] \times [c, d]$

Integral Iterada



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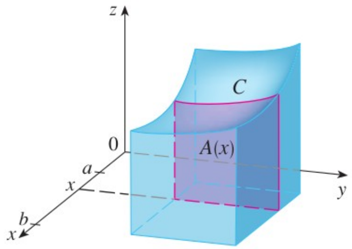

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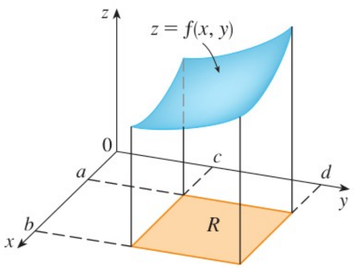

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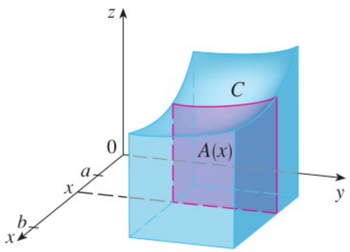

$A(x) = \int_c^d f(x, y)\, dy$

Integral Iterada

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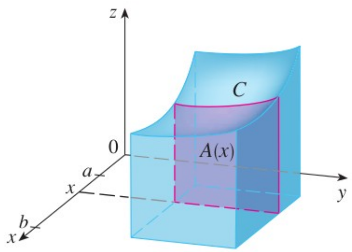
$$\int_a^b A(x)\, dx = \int_a^b \left[\int_c^d f(x, y)\, dy \right] dx$$



Integral Iterada

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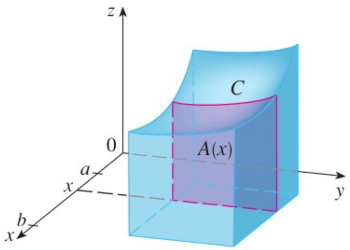
$$\int_a^b A(x)\, dx = \int_a^b \left[\int_c^d f(x, y)\, dy \right] dx$$

$$\int_a^b \int_c^d f(x, y)\, dy\, dx = \int_a^b \left[\int_c^d f(x, y)\, dy \right] dx$$

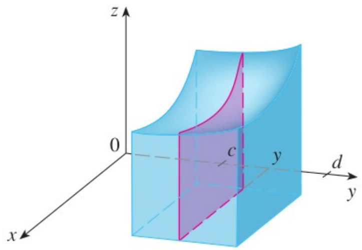


Exemplo: $\int_0^3 \int_1^2 x^2 y\, dy\, dx$

Integral Iterada

Analogamente

$$\int_c^d \int_a^b f(x, y)\, dx\, dy = \int_c^d \left[\int_a^b f(x, y)\, dx \right] dy$$



Exemplo: $\int_0^3 \int_1^2 x^2 y\, dy\, dx$

$$\int_1^2 x^2 y\, dy = \left[x^2 \frac{y^2}{2} \right]_{y=1}^{y=2} = x^2\left(\frac{2^2}{2}\right) - x^2\left(\frac{1^2}{2}\right) = \tfrac{3}{2}x^2$$

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$$\int_0^3 \int_1^2 x^2 y \, dy \, dx = \int_0^3 \left[\int_1^2 x^2 y \, dy \right] dx$$

$$= \int_0^3 \tfrac{3}{2} x^2 \, dx = \frac{x^3}{2} \Big]_0^3 = \frac{27}{2}$$

Teorema de Fubini: Se f é contínua em $R = \{(x, y) \mid a \leqslant x \leqslant b, c \leqslant y \leqslant d\}$, então

$$\iint_R f(x, y) \, dA = \int_a^b \int_c^d f(x, y) \, dy \, dx = \int_c^d \int_a^b f(x, y) \, dx \, dy$$

Exemplo: $\iint_R (x - 3y^2) \, dA$, $R = \{(x, y) \mid 0 \leqslant x \leqslant 2, 1 \leqslant y \leqslant 2\}$

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$$\iint_R (x - 3y^2) \, dA = \int_0^2 \int_1^2 (x - 3y^2) \, dy \, dx = \int_0^2 \left[xy - y^3 \right]_{y=1}^{y=2} dx$$

$$= \int_0^2 (x - 7) \, dx = \frac{x^2}{2} - 7x \Big]_0^2 = -12$$

Exemplo: $\iint_R (x - 3y^2)\, dA,\ R = \{(x, y) \mid 0 \leq x \leq 2, 1 \leq y \leq 2\}$

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$$\iint_R (x - 3y^2)\, dA = \int_1^2 \int_0^2 (x - 3y^2)\, dx\, dy$$

$$= \int_1^2 \left[\frac{x^2}{2} - 3xy^2\right]_{x=0}^{x=2} dy$$

$$= \int_1^2 (2 - 6y^2)\, dy = 2y - 2y^3\Big]_1^2 = -12$$

Exemplo: $\iint_R y \sin(xy)\, dA,\ R = [1, 2] \times [0, \pi]$

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Exemplo: $\iint_R y \sin(xy)\, dA,\ R = [1, 2] \times [0, \pi]$

$$\iint_R y \sin(xy)\, dA = \int_1^2 \int_0^\pi y \sin(xy)\, dy\, dx$$

$$u = y \qquad dv = \sin(xy)\, dy$$

$$du = dy \qquad v = -\frac{\cos(xy)}{x}$$

Exemplo: $\iint_R y\sin(xy)\,dA$, $R=[1,2]\times[0,\pi]$

$$\iint\limits_R y\sin(xy)\,dA = \int_1^2\int_0^\pi y\sin(xy)\,dy\,dx$$

$$u=y \qquad dv=\sin(xy)\,dy$$
$$du=dy \qquad v=-\frac{\cos(xy)}{x}$$

$$\int_0^\pi y\sin(xy)\,dy = -\frac{y\cos(xy)}{x}\Bigg]_{y=0}^{y=\pi} + \frac{1}{x}\int_0^\pi \cos(xy)\,dy$$
$$= -\frac{\pi\cos\pi x}{x} + \frac{1}{x^2}\left[\sin(xy)\right]_{y=0}^{y=\pi}$$
$$= -\frac{\pi\cos\pi x}{x} + \frac{\sin\pi x}{x^2}$$

$$\int\left(-\frac{\pi\cos\pi x}{x}\right)dx = -\frac{\sin\pi x}{x} - \int\frac{\sin\pi x}{x^2}\,dx$$

$$\int\left(-\frac{\pi\cos\pi x}{x}+\frac{\sin\pi x}{x^2}\right)dx = -\frac{\sin\pi x}{x}$$

Exemplo: $\iint_R y\sin(xy)\,dA$, $R=[1,2]\times[0,\pi]$

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$$= -\frac{\pi\cos\pi x}{x} + \frac{\sin\pi x}{x^2}$$

$$\int\left(-\frac{\pi\cos\pi x}{x}\right)dx = -\frac{\sin\pi x}{x} - \int\frac{\sin\pi x}{x^2}\,dx$$

$$u=-1/x \qquad dv=\pi\cos\pi x\,dx$$
$$du=dx/x^2 \qquad v=\sin\pi x$$

$$\int\left(-\frac{\pi\cos\pi x}{x}\right)dx = -\frac{\sin\pi x}{x} - \int\frac{\sin\pi x}{x^2}\,dx$$

$$\int\left(-\frac{\pi\cos\pi x}{x}+\frac{\sin\pi x}{x^2}\right)dx = -\frac{\sin\pi x}{x}$$

$$\int_1^2\int_0^\pi y\sin(xy)\,dy\,dx = \left[-\frac{\sin\pi x}{x}\right]_1^2$$
$$= -\frac{\sin 2\pi}{2} + \sin\pi = 0$$

Solução alternativa:

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$$\iint_R y \sin(xy)\, dA = \int_0^{\pi} \int_1^{2} y \sin(xy)\, dx\, dy = \int_0^{\pi} \Big[-\cos(xy)\Big]_{x=1}^{x=2} dy$$

$$= \int_0^{\pi} (-\cos 2y + \cos y)\, dy$$

$$= -\tfrac{1}{2} \sin 2y + \sin y\Big]_0^{\pi} = 0$$

Suponha $f(x, y) = g(x)h(y)$

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$$= \int_c^d \left[h(y) \left(\int_a^b g(x)\, dx \right) \right] dy = \int_a^b g(x)\, dx \int_c^d h(y)\, dy$$

Exemplo: $R = [0, \pi/2] \times [0, \pi/2]$

$$\iint_R \sin x \cos y\, dA = \int_0^{\pi/2} \sin x\, dx \int_0^{\pi/2} \cos y\, dy$$

$$= \left[-\cos x \right]_0^{\pi/2} \left[\sin y \right]_0^{\pi/2} = 1 \cdot 1 = 1$$

Suponha $f(x, y) = g(x)h(y)$

$$\iint_R f(x, y)\, dA = \int_c^d \int_a^b g(x)h(y)\, dx\, dy = \int_c^d \left[\int_a^b g(x)h(y)\, dx \right] dy$$

$$= \int_c^d \left[h(y) \left(\int_a^b g(x)\, dx \right) \right] dy = \int_a^b g(x)\, dx \int_c^d h(y)\, dy$$

$$\iint_R g(x)\, h(y)\, dA = \int_a^b g(x)\, dx \int_c^d h(y)\, dy \qquad R = [a, b] \times [c, d]$$