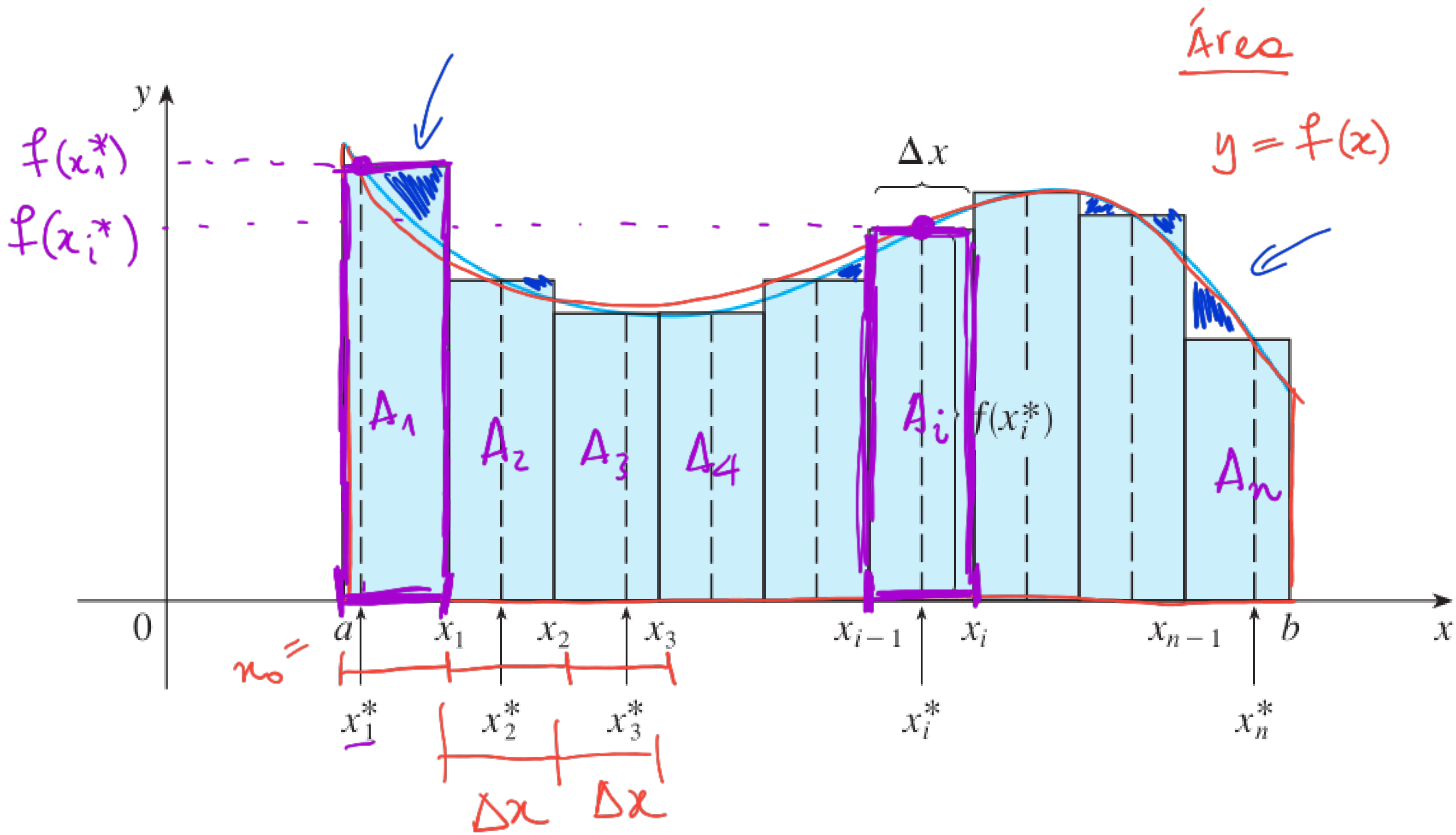
Cálculo III

Integral dupla

Prof. Adriano Barbosa



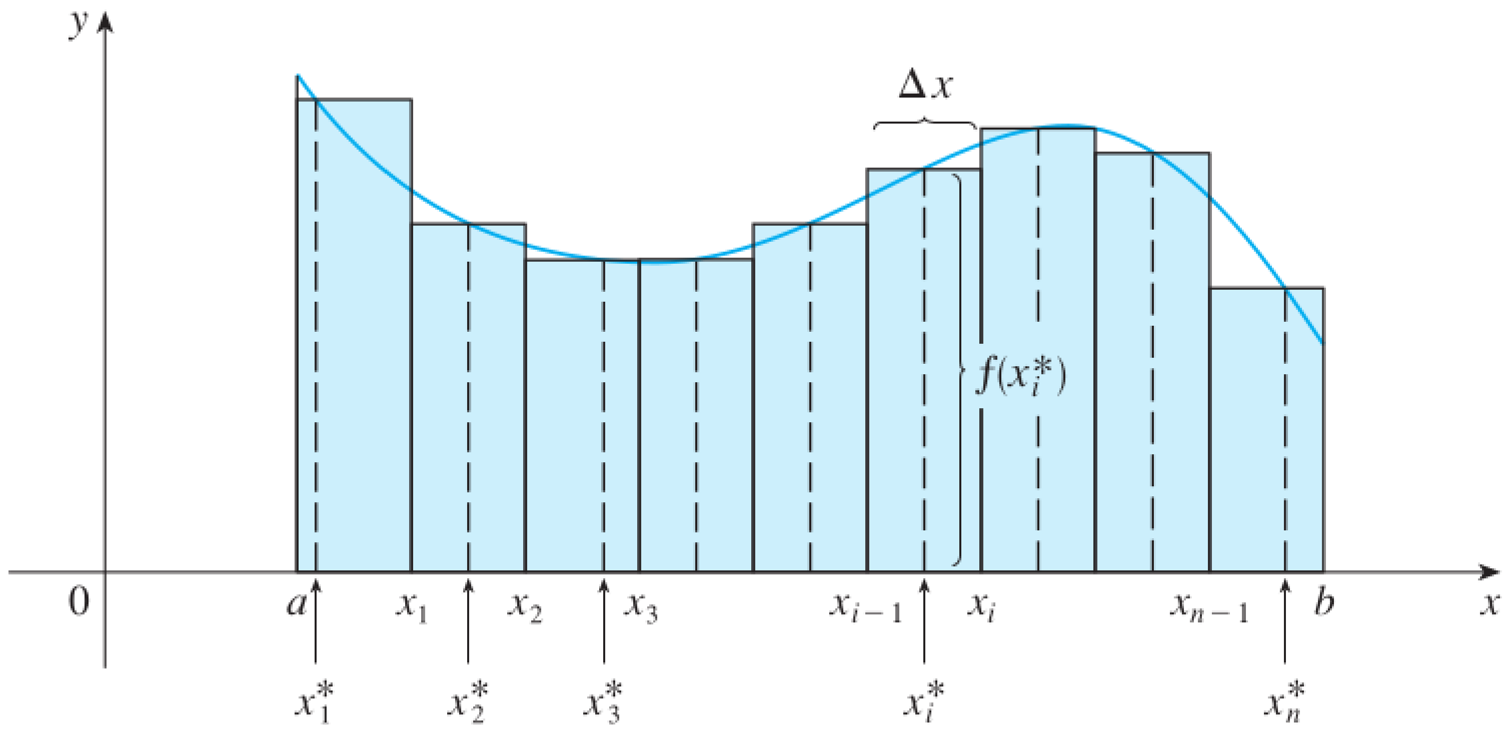
$$\Delta x = \frac{b-a}{n}$$

$$A \approx A_1 + A_2 + A_3 + \dots + A_i + \dots + A_n$$

$$= \Delta x \cdot f(x_1^*) + \Delta x \cdot f(x_2^*) + \dots + \Delta x f(x_i^*) + \dots + \Delta x f(x_n^*)$$

$$x_i^* \in [x_{i-1}, x_i]$$

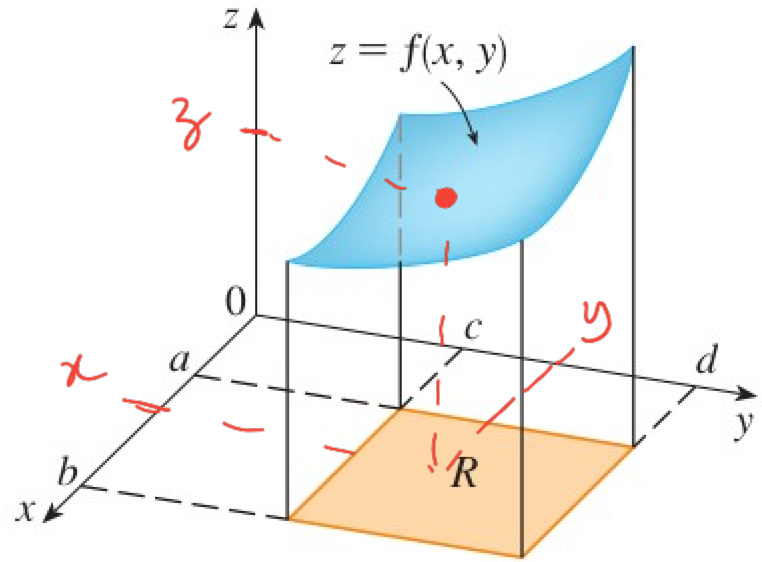
$$\int_a^b f(x)dx = A = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*) \Delta x \quad \text{(Soma Riemann)}$$



$$A \approx \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

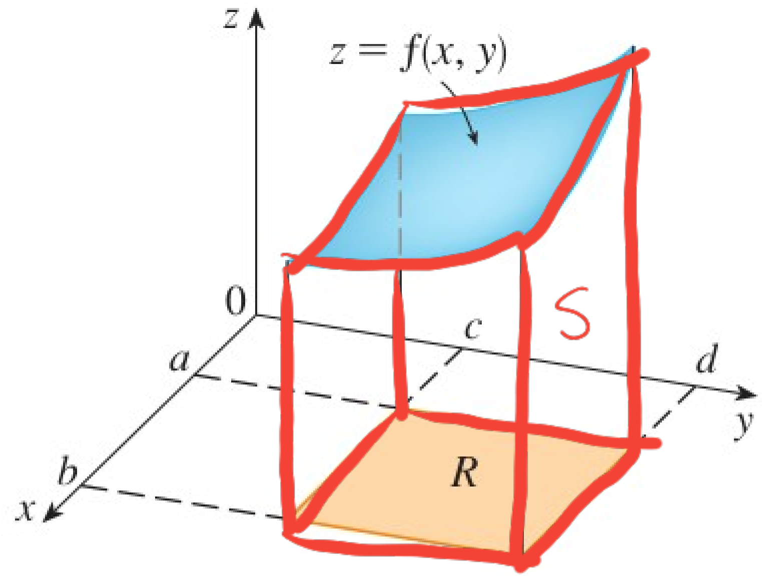
h b

$$\int_a^b f(x)\,dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$



$R = [a,b] \times [c,d]$

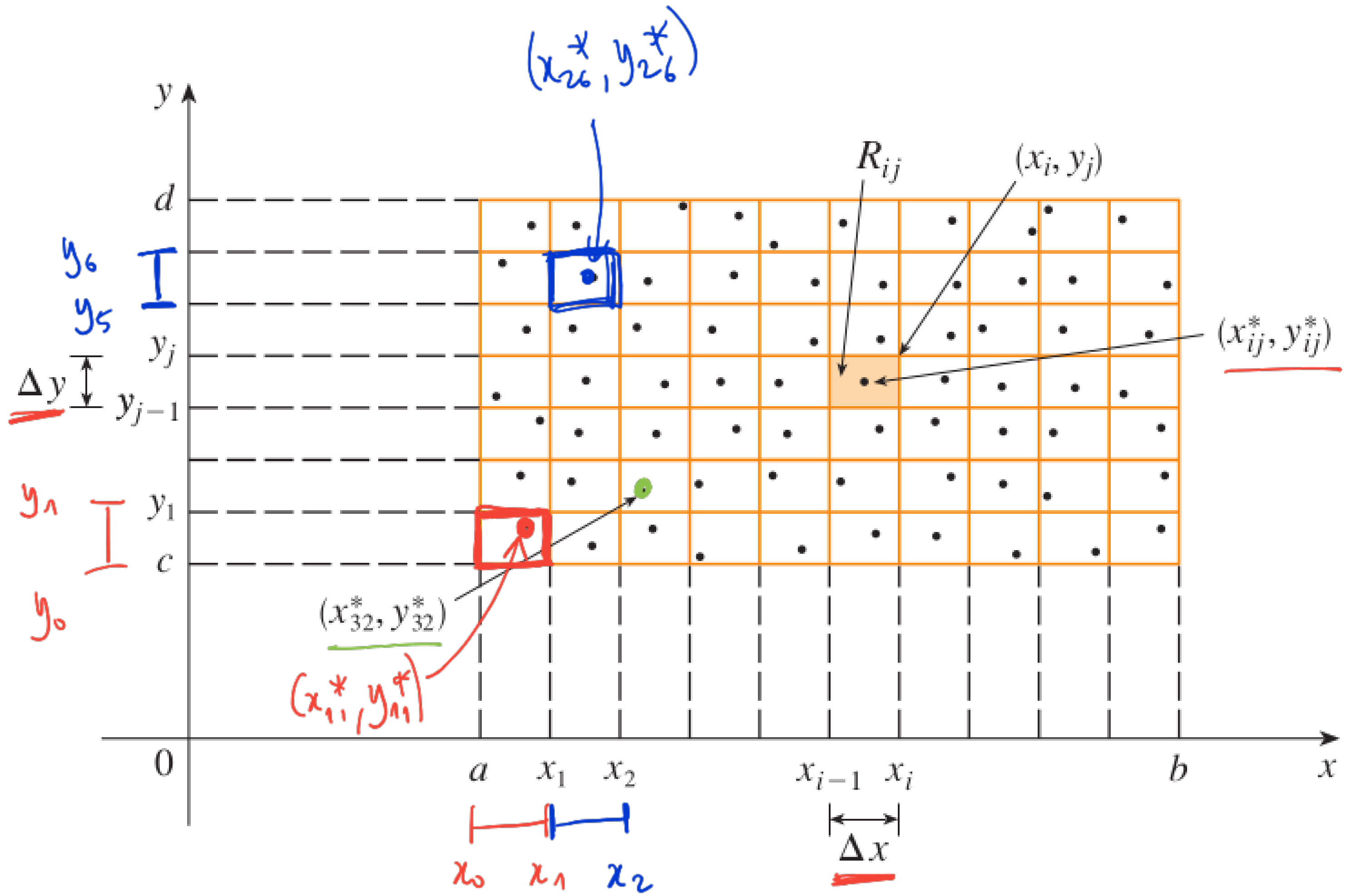
$f(x, y) \geq 0$



$f(x, y) \geqslant 0$

$$R = [a, b] \times [c, d] = \{(x, y) \in \mathbb{R}^2 \mid a \leqslant x \leqslant b,\ c \leqslant y \leqslant d\}$$

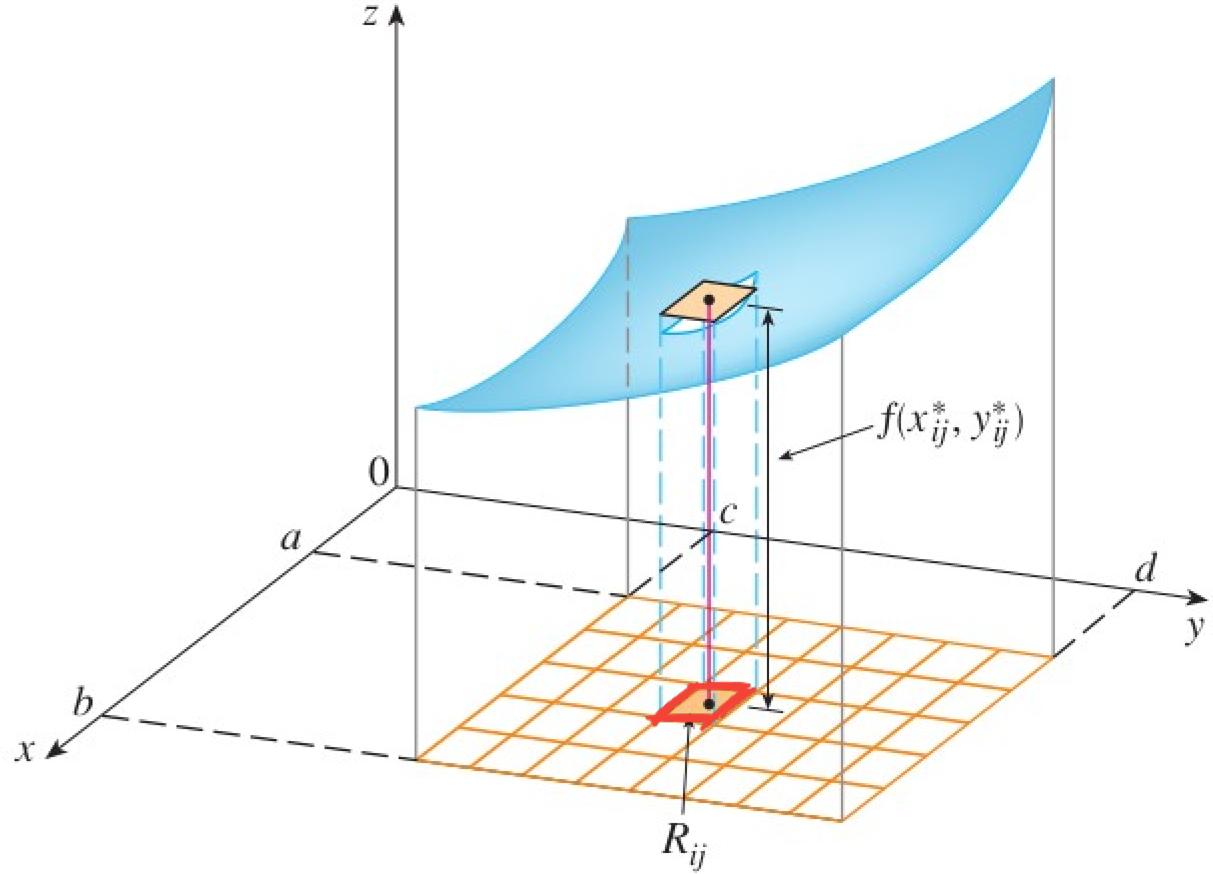
$$S = \{(x, y, z) \in \mathbb{R}^3 \mid 0 \leqslant z \leqslant f(x, y),\ (x, y) \in R\}$$



$$\Delta x = (b - a)/m \qquad \Delta y = (d - c)/n$$

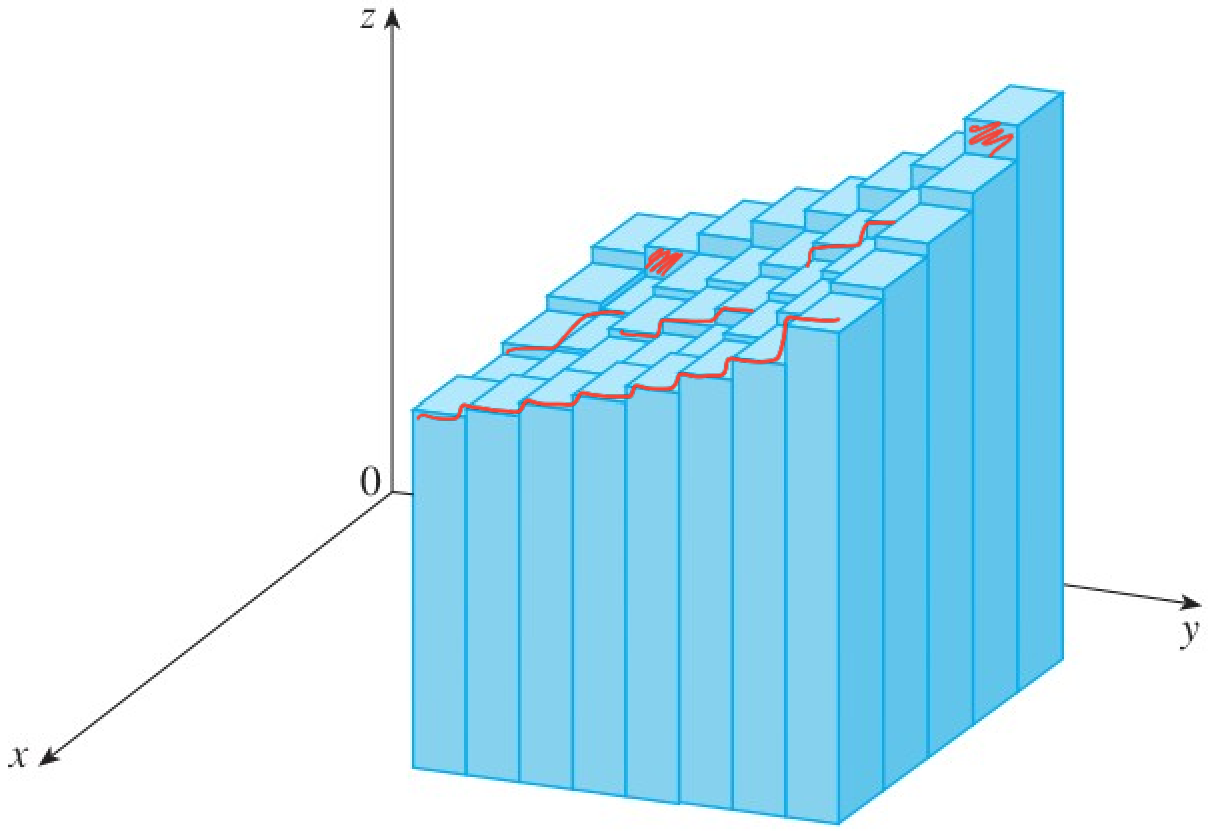
$$(x^*_{ij}, y^*_{ij}) \in R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] = \{(x, y) \mid x_{i-1} \leqslant x \leqslant x_i,\ y_{j-1} \leqslant y \leqslant y_j\}$$

$$(x^*_{11}, y^*_{11}) \in R_{11} = [x_0, x_1] \times [y_0, y_1] \qquad R_{26} = [x_1, x_2] \times [y_5, y_6]$$



$$\Delta A = \Delta x \, \Delta y$$

$$f(x_{ij}^*, y_{ij}^*) \, \Delta A$$

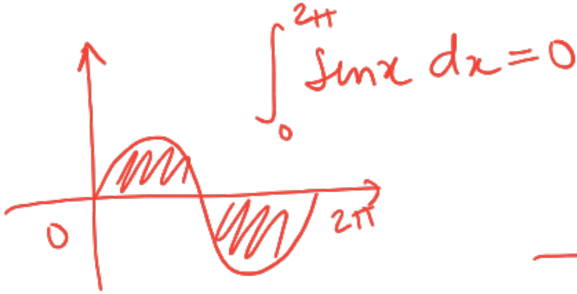


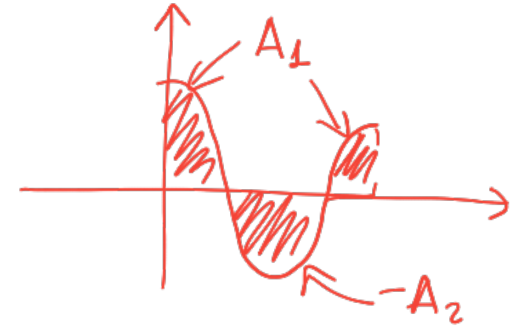
$$V \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\, \Delta A$$

$$V = \lim_{m,\, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\, \Delta A$$

Integral dupla:

$$\iint_R f(x, y)\, dA = \lim_{m, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\, \Delta A$$



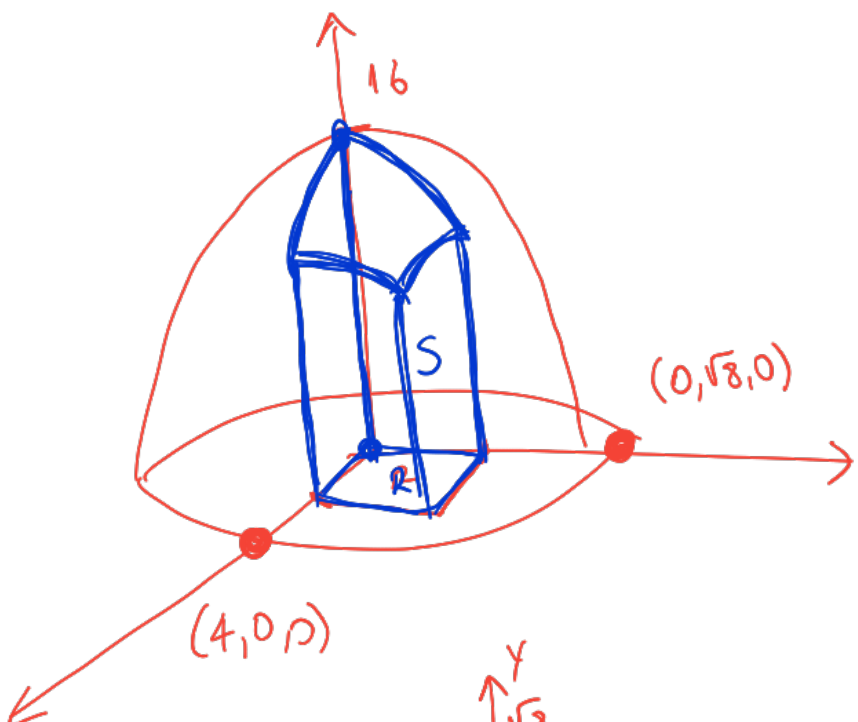


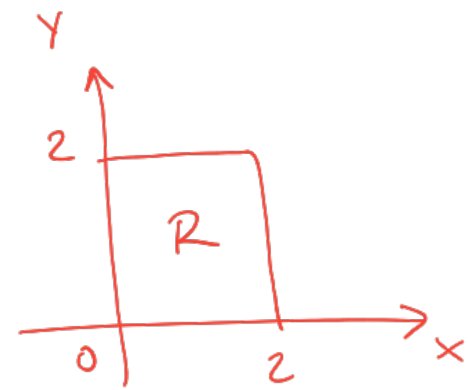
Se $f(x, y) \geqslant 0$, o volume do sólido acima da região *R* e abaixo gráfico da função é dado por:

$$V = \iint_R f(x, y)\, dA$$

Exemplo: Estime ~~a área~~ o volume do sólido acima da região R e abaixo do gráfico da função abaixo

$$R = [0,2] \times [0,2] \qquad f(x,y) = 16 - \underbrace{x^2}_{\leq 0} - \underbrace{2y^2}_{\leq 0}$$

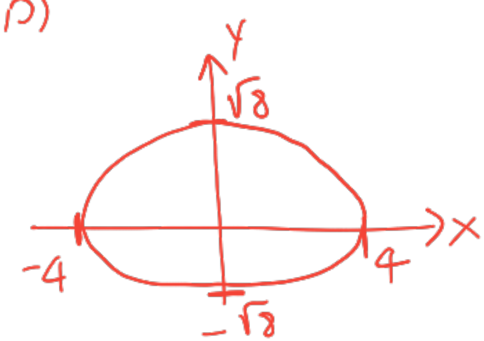




$$16 - x^2 - 2y^2 = 0$$

$$x^2 + 2y^2 = 16$$

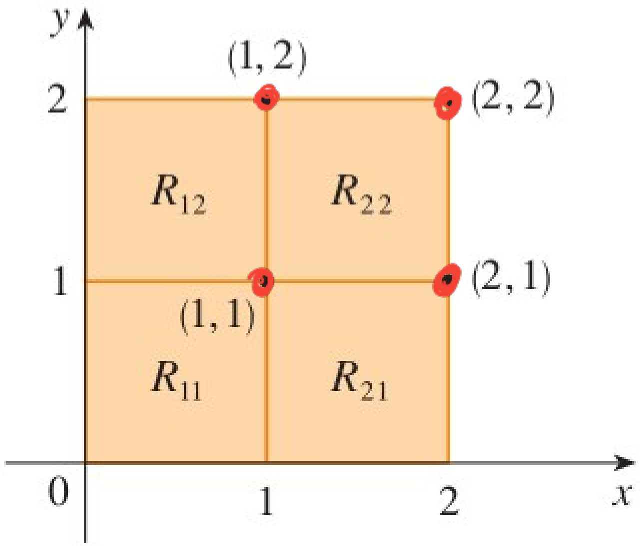
$$\frac{x^2}{16} + \frac{y^2}{8} = 1 \Rightarrow \frac{x^2}{4^2} + \frac{y^2}{(\sqrt{8})^2} = 1$$



$$\sqrt{8} = 2\sqrt{2} > 2$$

Exemplo: Estime ~~a área~~ o volume do sólido acima da região R e abaixo do gráfico da função abaixo

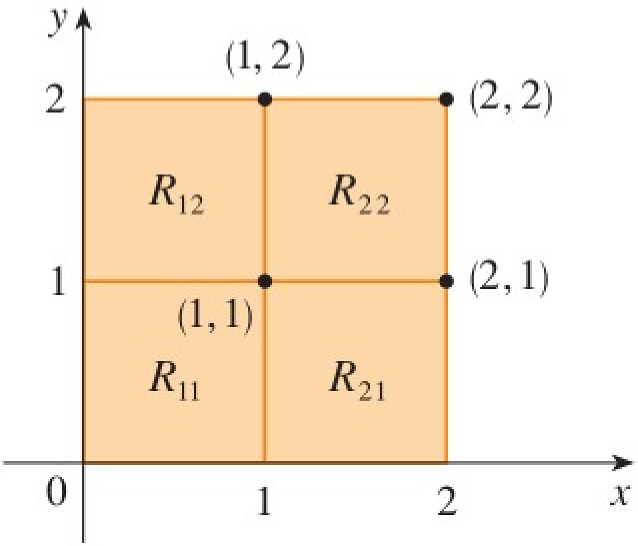
$$R = [0, 2] \times [0, 2] \qquad f(x, y) = 16 - x^2 - 2y^2$$



o volume

Exemplo: Estime ~~a área~~ do sólido acima da região *R* e abaixo do gráfico da função abaixo

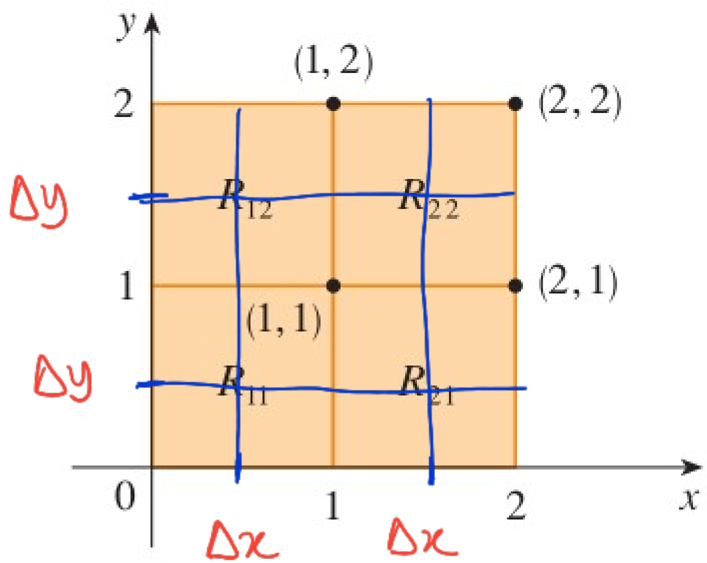
$R = [0, 2] \times [0, 2]$ $f(x, y) = 16 - x^2 - 2y^2$



$$V \approx \sum_{i=1}^{2} \sum_{j=1}^{2} f(x_i, y_j)\, \Delta A$$

Exemplo: Estime ~~a área~~ o volume do sólido acima da região R e abaixo do gráfico da função abaixo

$$R = [0, 2] \times [0, 2] \qquad f(x, y) = 16 - x^2 - 2y^2$$

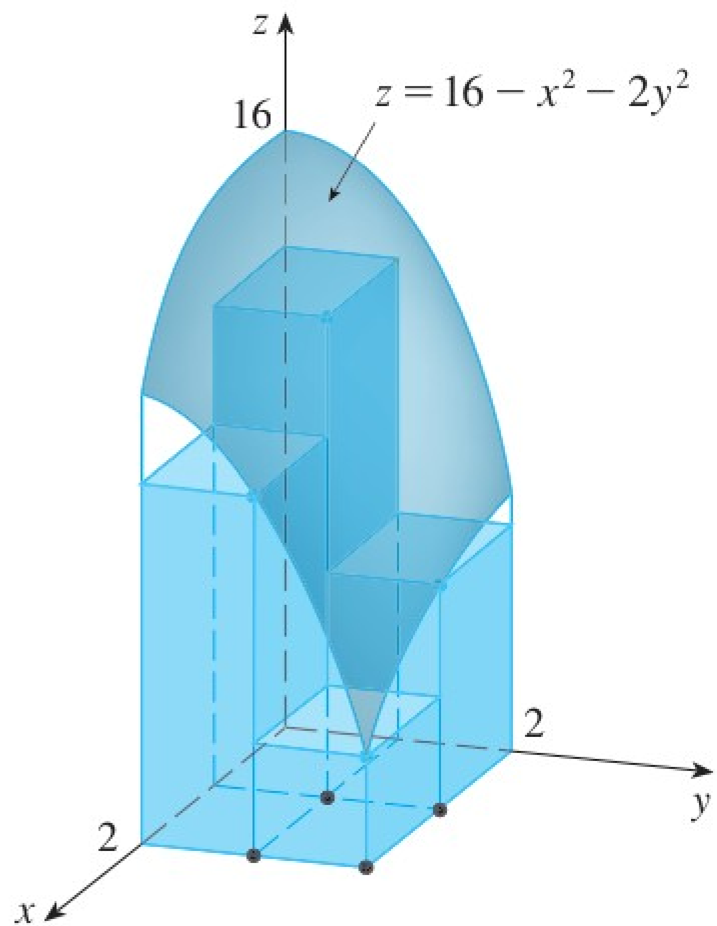


$$\Delta x = \frac{2-0}{2} = 1$$

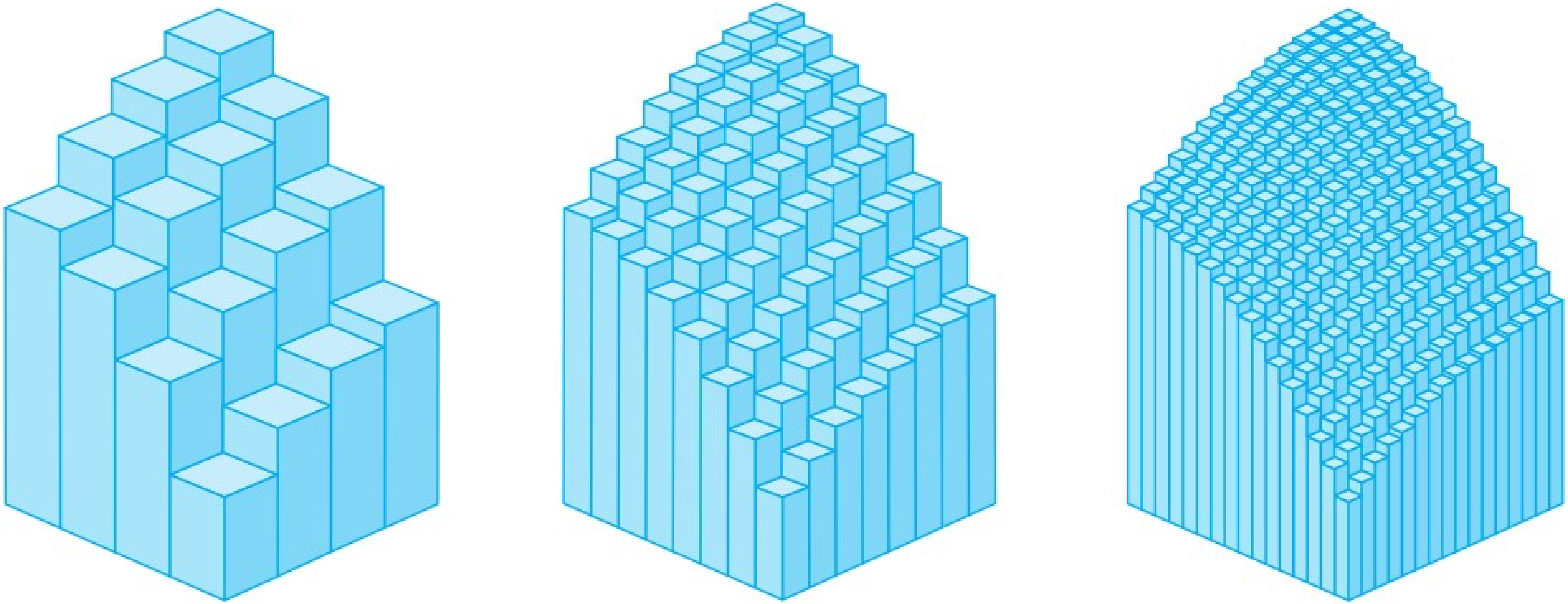
$$\Delta y = \frac{2-0}{2} = 1$$

$$V \approx \sum_{i=1}^{2} \sum_{j=1}^{2} f(x_i, y_j)\,\Delta A = f(1, 1)\,\Delta A + f(1, 2)\,\Delta A + f(2, 1)\,\Delta A + f(2, 2)\,\Delta A$$

$$= 13(1) + 7(1) + 10(1) + 4(1) = 34$$

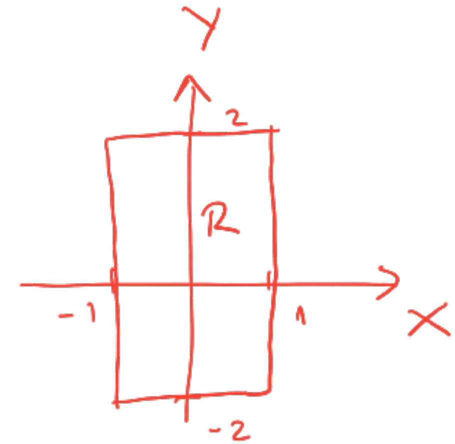


z
16
z = 16 − x² − 2y²
2
y
2
x



(a) $m = n = 4$, $V \approx 41.5$ (b) $m = n = 8$, $V \approx 44.875$ (c) $m = n = 16$, $V \approx 46.46875$

Exemplo: $R = \{(x, y) \mid -1 \leqslant x \leqslant 1, -2 \leqslant y \leqslant 2\}$



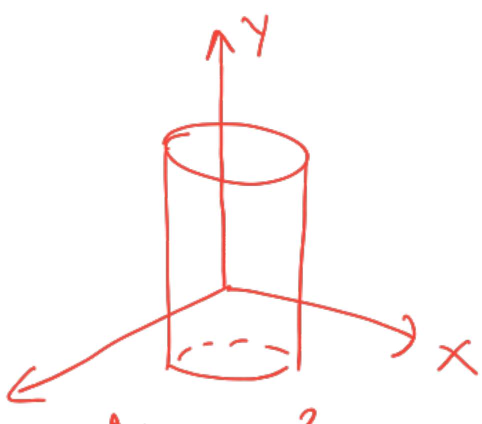
$f(x,y) = \sqrt{1-x^2} \geqslant 0$ $\quad 2\pi = V = \iint_R \sqrt{1 - x^2}\, dA$

$= \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\, \Delta A$

$z = \sqrt{1-x^2} \Leftrightarrow z^2 = 1-x^2$ e $z \geqslant 0$

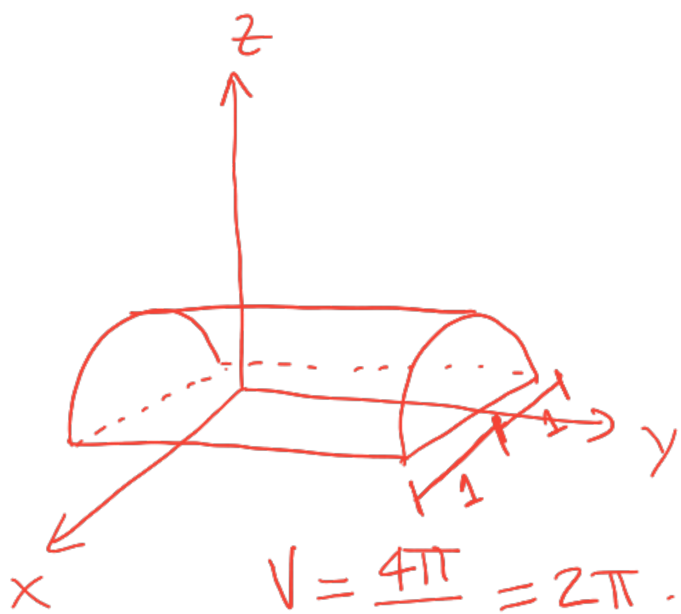
$\Leftrightarrow x^2 + z^2 = 1$ e $z \geqslant 0$

↑ cilindro



$A_b = \pi \cdot 1^2 = \pi$

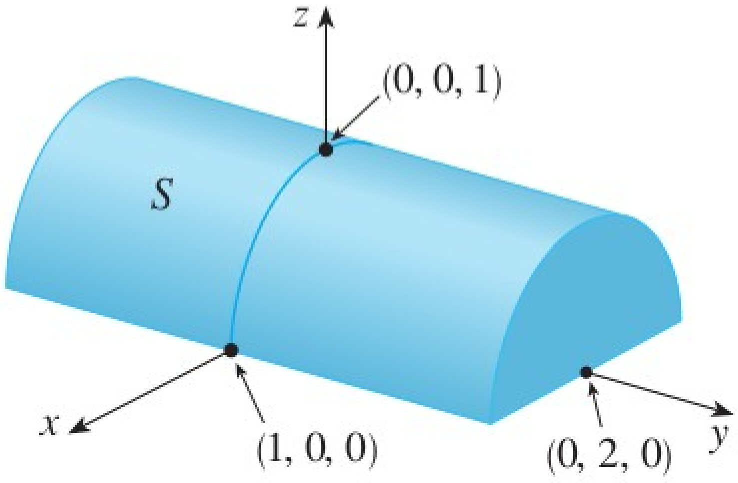
$V = \pi \cdot 4$



$V = \frac{4\pi}{2} = 2\pi.$

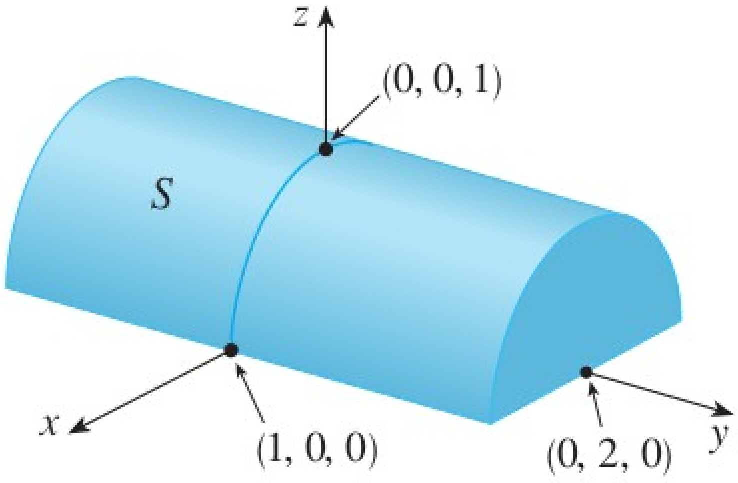
Exemplo: $R = \{(x, y) \mid -1 \leqslant x \leqslant 1, -2 \leqslant y \leqslant 2\}$

$$\iint_R \sqrt{1 - x^2}\, dA$$



Exemplo: $R = \{(x, y) \mid -1 \leqslant x \leqslant 1, -2 \leqslant y \leqslant 2\}$

$$\iint_R \sqrt{1 - x^2}\, dA$$



$$\iint_R \sqrt{1 - x^2}\, dA = \tfrac{1}{2}\pi(1)^2 \times 4 = 2\pi$$

Propriedades

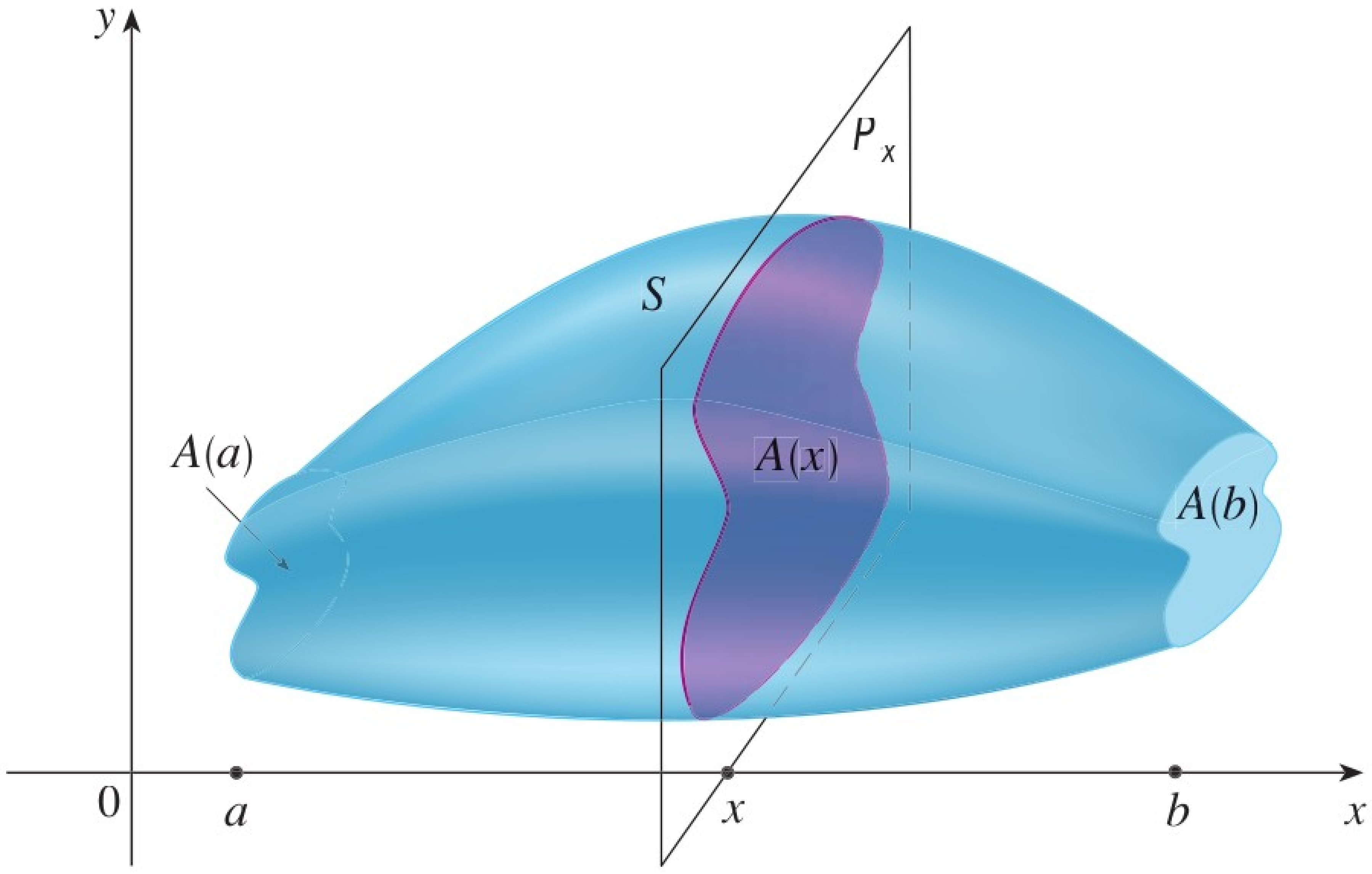
$$\iint_R [f(x, y) + g(x, y)]\, dA = \iint_R f(x, y)\, dA + \iint_R g(x, y)\, dA$$

$$\iint_R c f(x, y)\, dA = c \iint_R f(x, y)\, dA$$

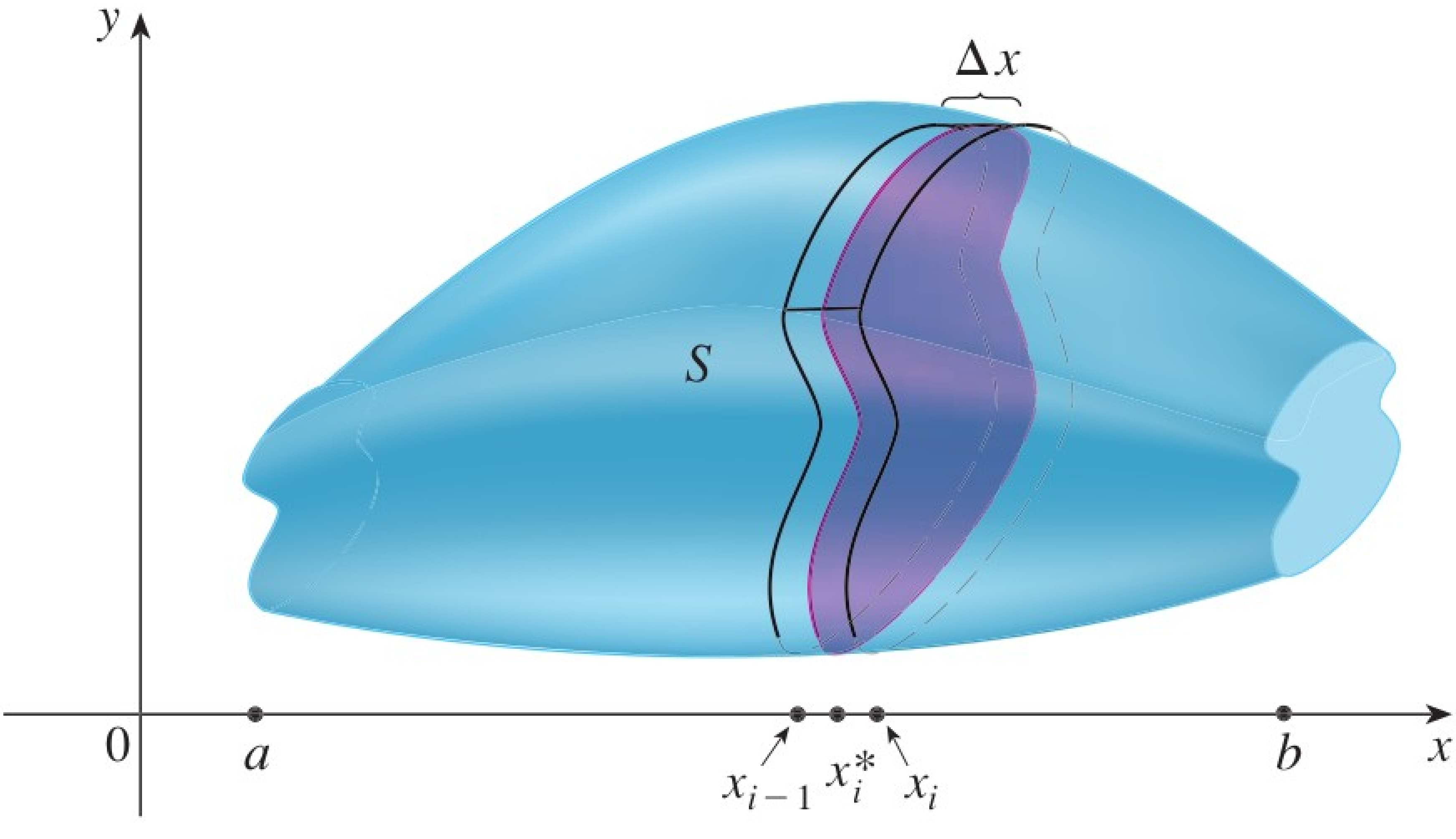
$$f(x, y) \geqslant g(x, y)$$

$$\iint_R f(x, y)\, dA \geqslant \iint_R g(x, y)\, dA$$

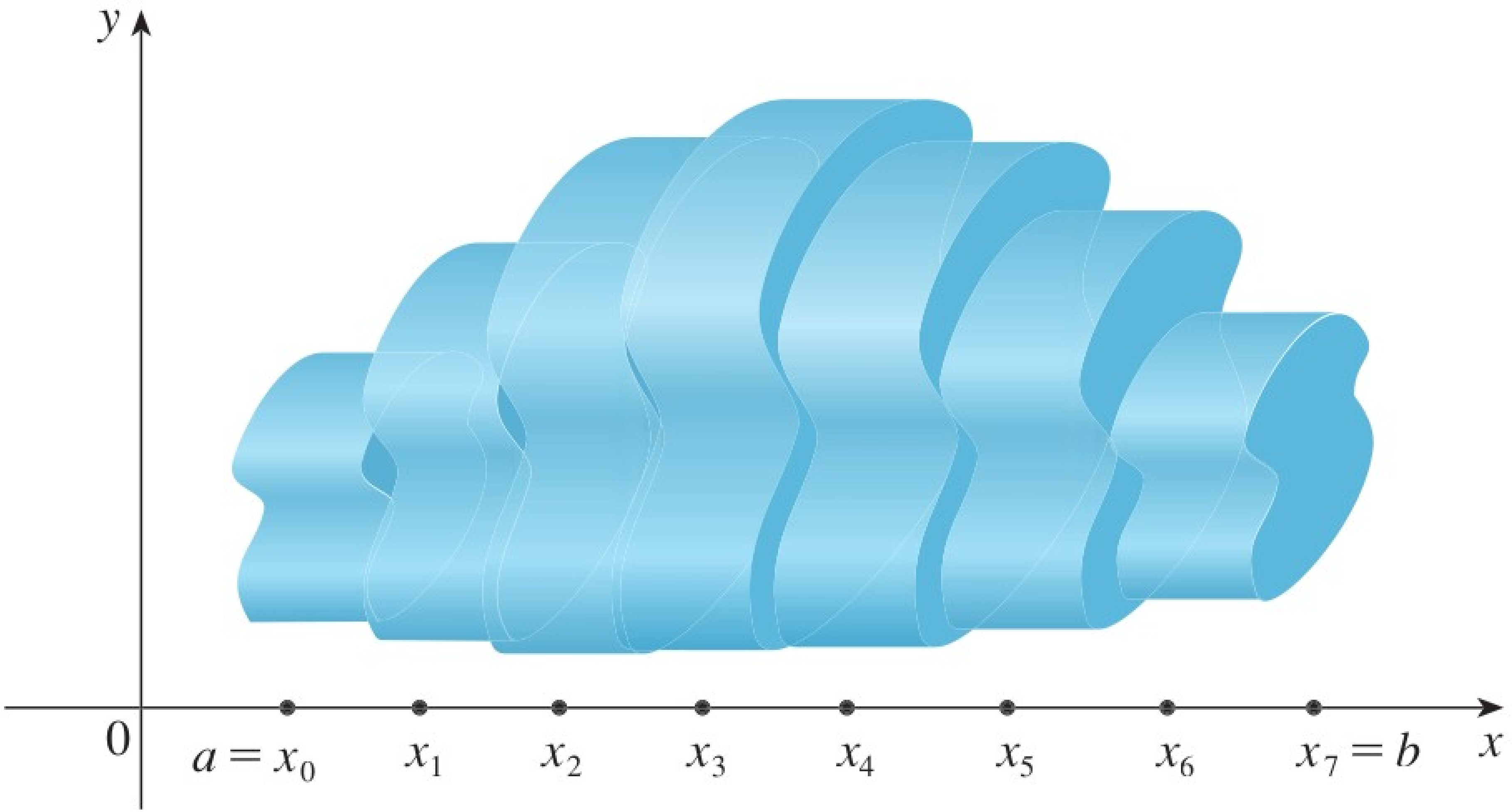
Integral Iterada



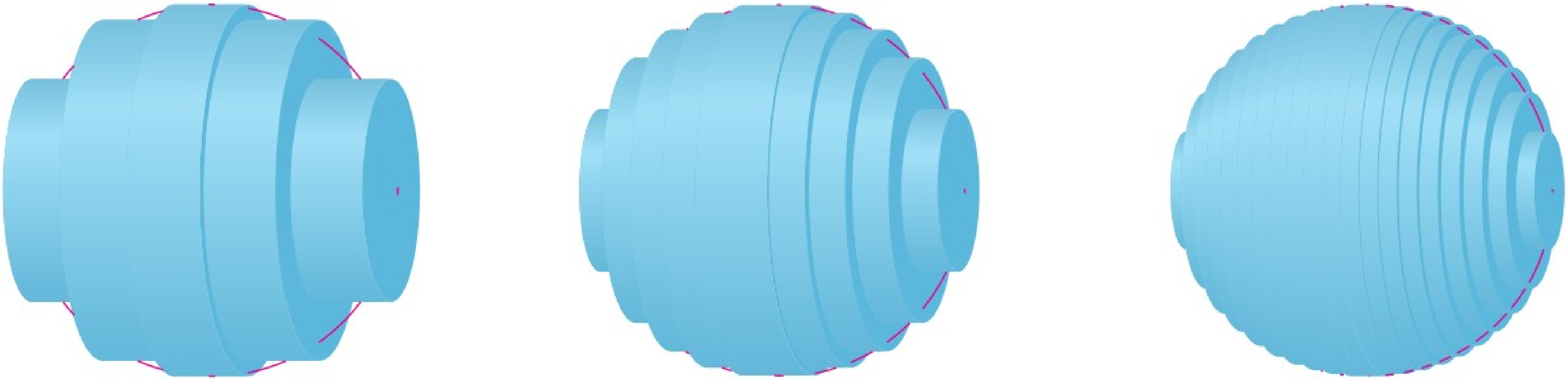
Integral Iterada



Integral Iterada



Integral Iterada

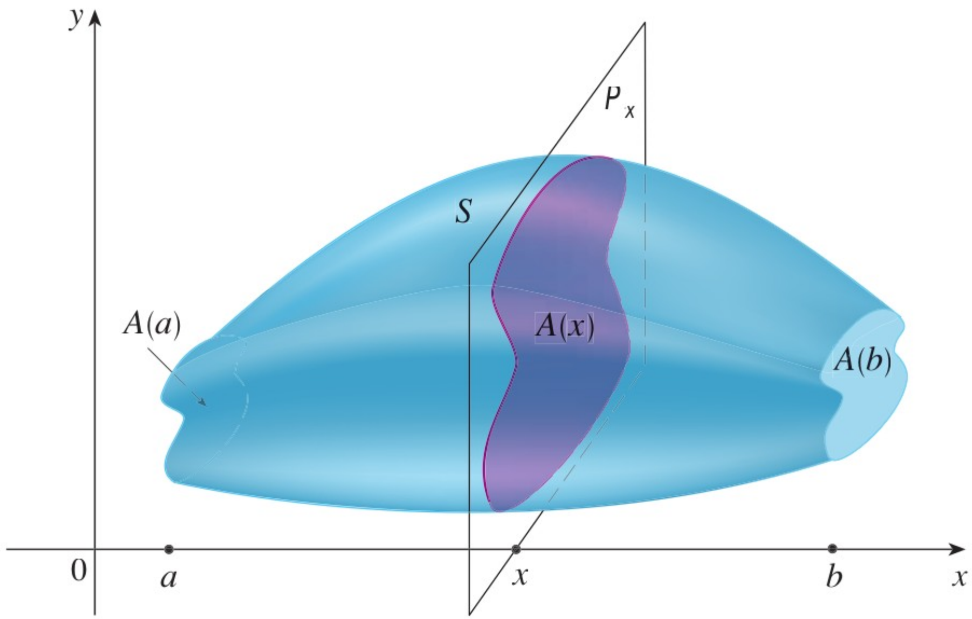


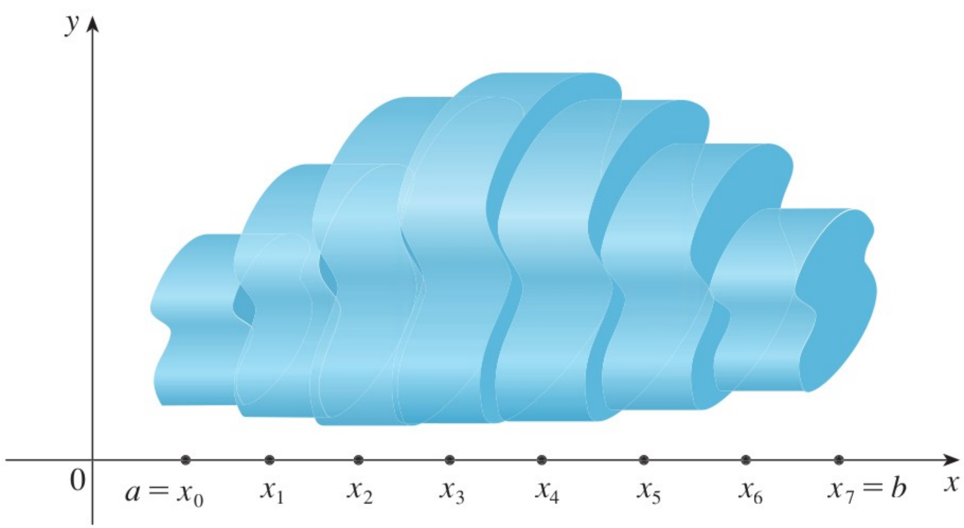
(a) Using 5 disks, $V \approx 4.2726$

(b) Using 10 disks, $V \approx 4.2097$

(c) Using 20 disks, $V \approx 4.1940$

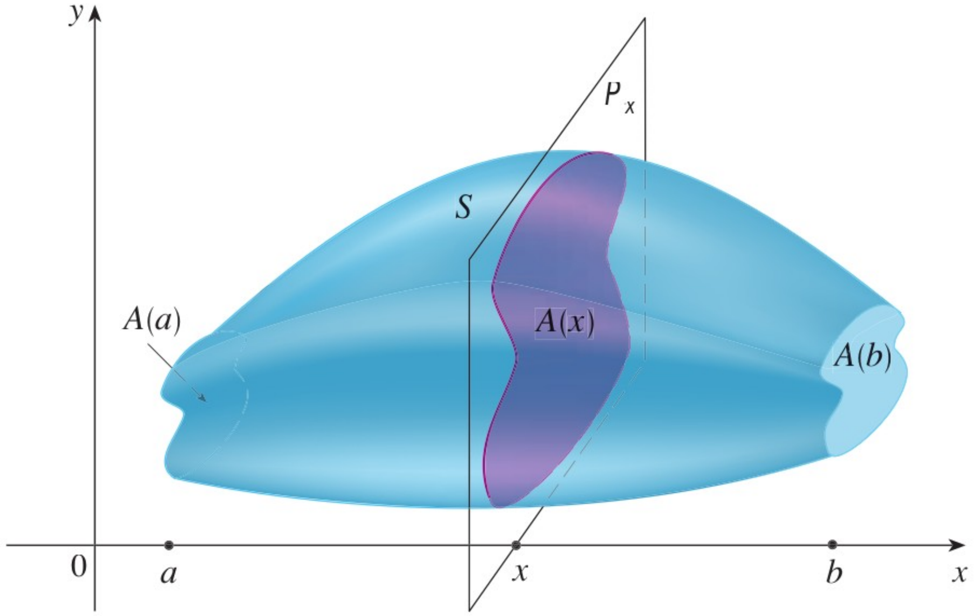
Integral Iterada





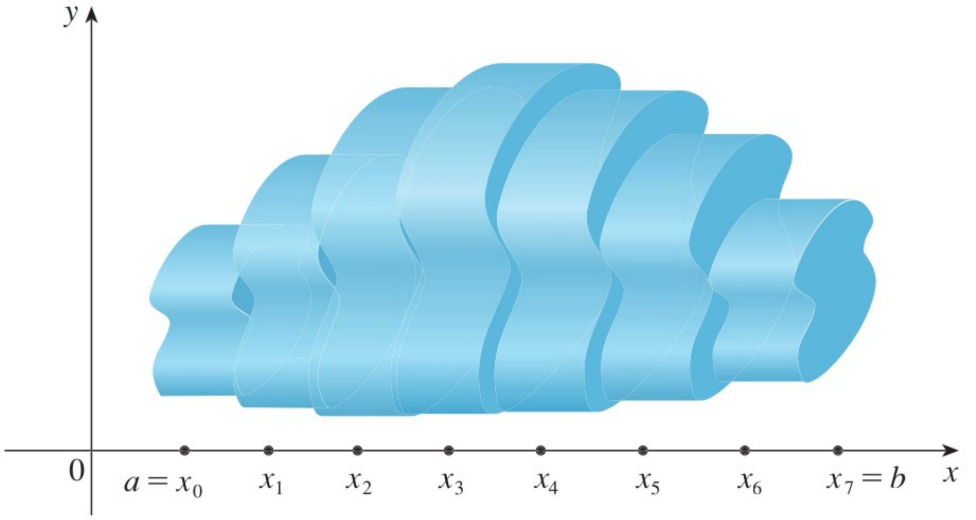
$$V(S_i) \approx A(x_i^*)\,\Delta x$$

Integral Iterada

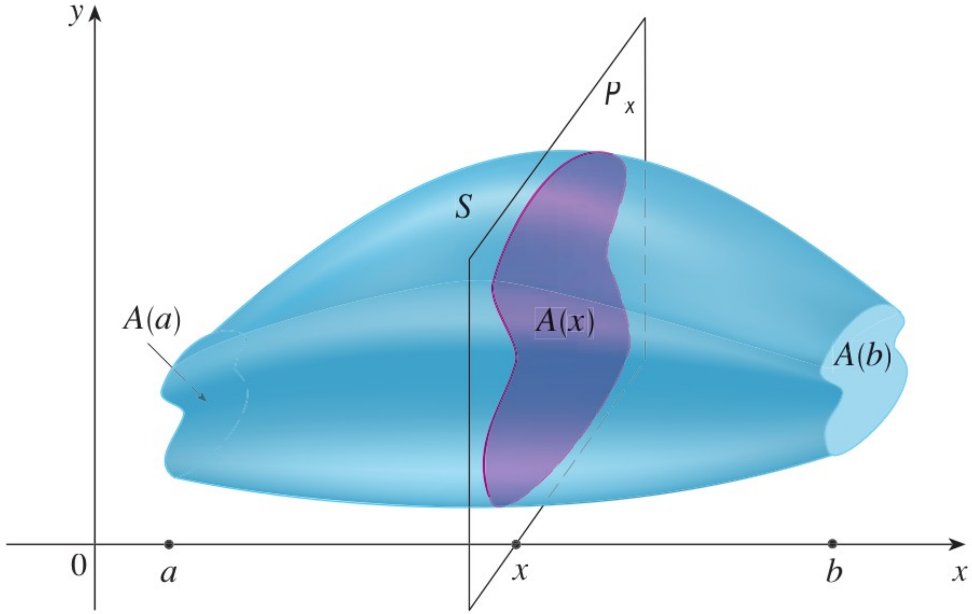


$$V(S_i) \approx A(x_i^*)\,\Delta x$$

$$V \approx \sum_{i=1}^{n} A(x_i^*)\,\Delta x$$

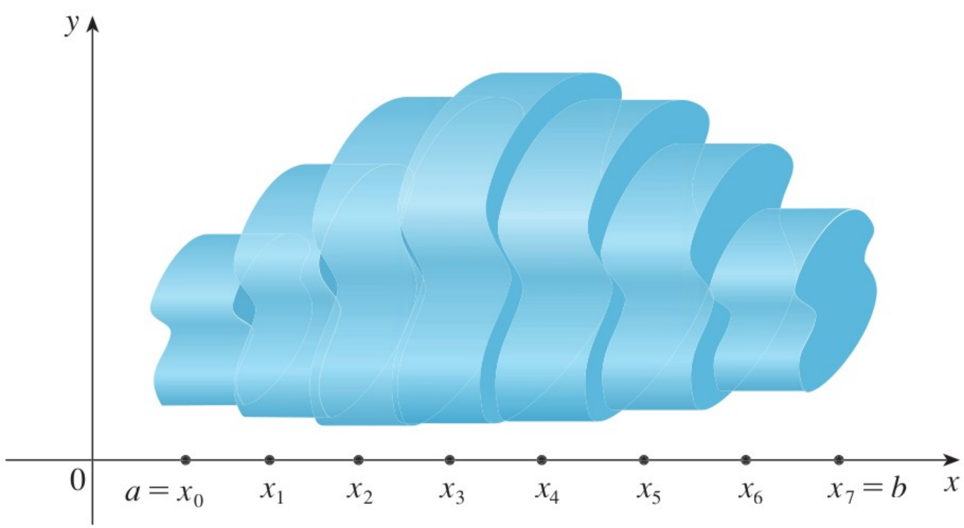


Integral Iterada



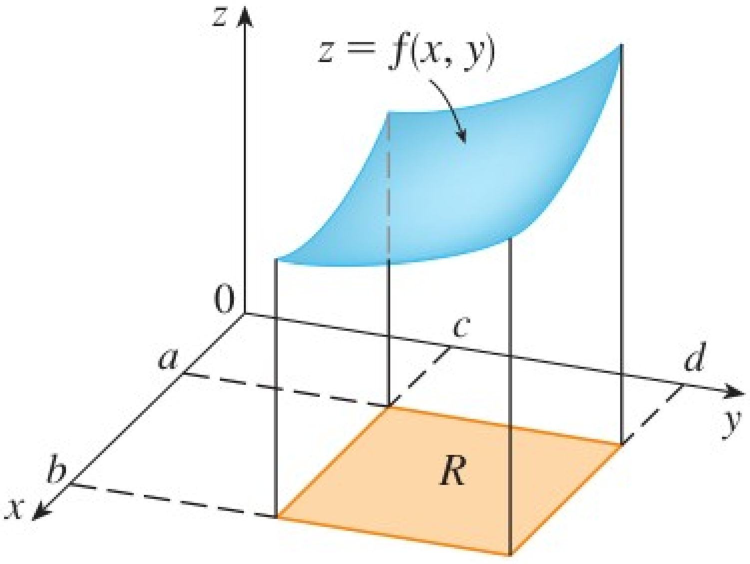
$$V(S_i) \approx A(x_i^*)\,\Delta x$$

$$V \approx \sum_{i=1}^{n} A(x_i^*)\,\Delta x$$



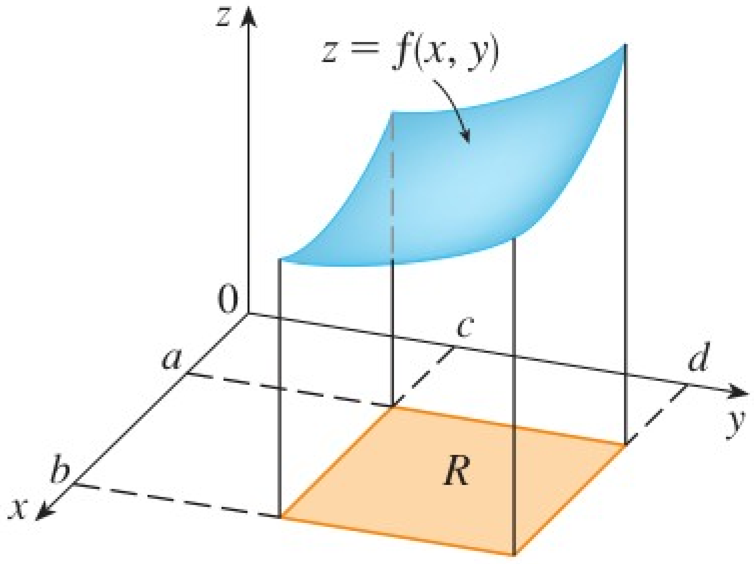
$$V = \lim_{n \to \infty} \sum_{i=1}^{n} A(x_i^*)\,\Delta x = \int_a^b A(x)\,dx$$

Integral Iterada

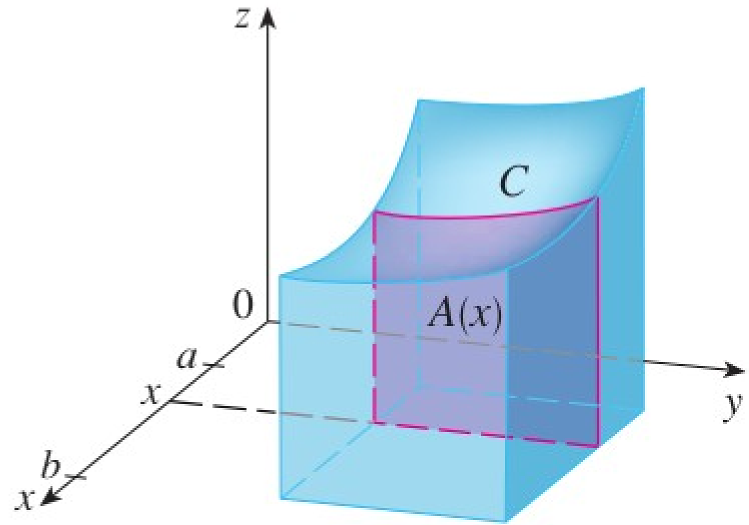


$$R = [a, b] \times [c, d]$$

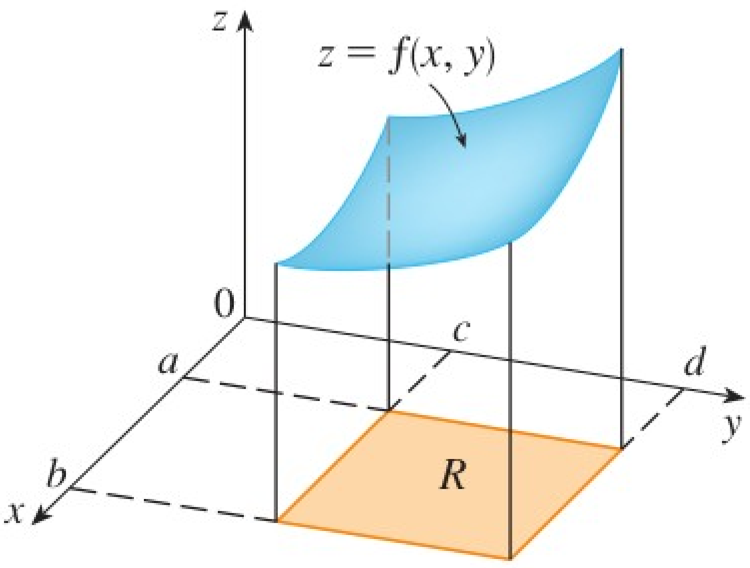
Integral Iterada



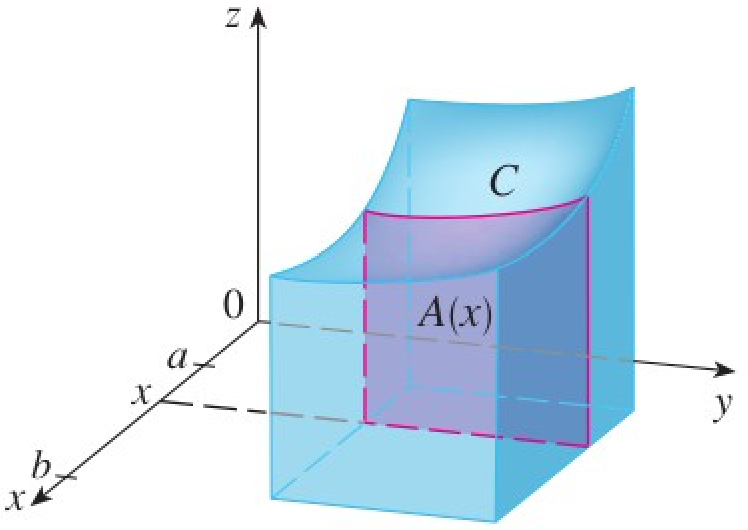
$R = [a, b] \times [c, d]$



Integral Iterada



$R = [a, b] \times [c, d]$

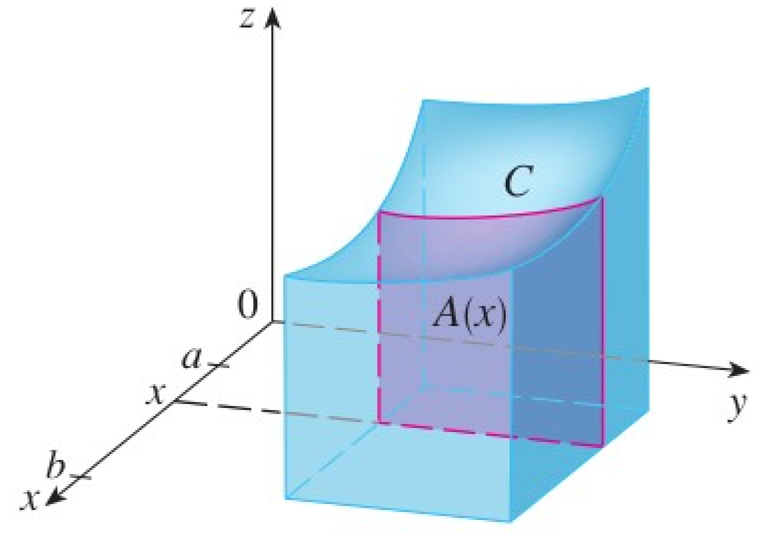


$A(x) = \int_c^d f(x, y)\, dy$

Integral Iterada

$$A(x) = \int_c^d f(x, y)\, dy$$

$$\int_a^b A(x)\, dx = \int_a^b \left[\int_c^d f(x, y)\, dy \right] dx$$

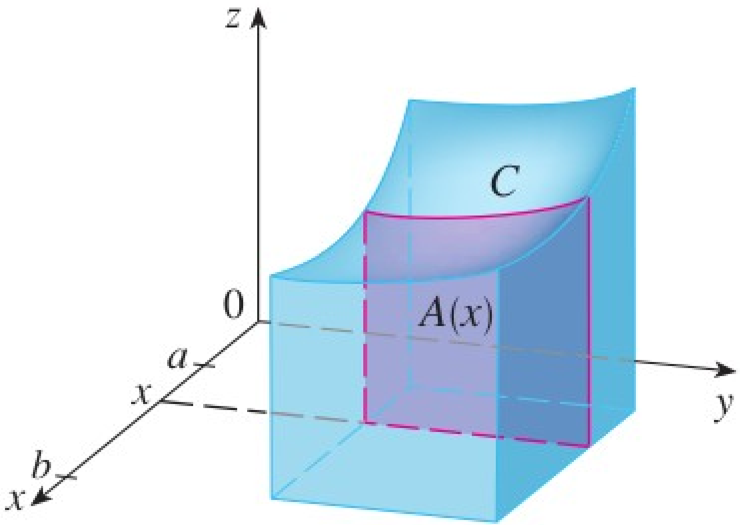


Integral Iterada

$$A(x) = \int_c^d f(x, y)\, dy$$

$$\int_a^b A(x)\, dx = \int_a^b \left[\int_c^d f(x, y)\, dy \right] dx$$

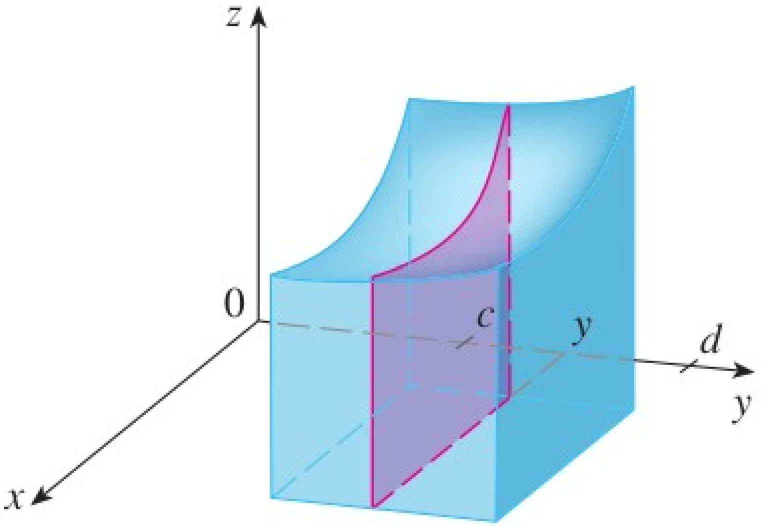
$$\int_a^b \int_c^d f(x, y)\, dy\, dx = \int_a^b \left[\int_c^d f(x, y)\, dy \right] dx$$



Integral Iterada

Analogamente

$$\int_c^d \int_a^b f(x, y)\, dx\, dy = \int_c^d \left[\int_a^b f(x, y)\, dx \right] dy$$



Exemplo: $\int_0^3 \int_1^2 x^2 y \, dy \, dx$

Exemplo: $\int_0^3 \int_1^2 x^2 y \, dy \, dx$

$$\int_1^2 x^2 y \, dy = \left[x^2 \frac{y^2}{2} \right]_{y=1}^{y=2} = x^2 \left(\frac{2^2}{2} \right) - x^2 \left(\frac{1^2}{2} \right) = \tfrac{3}{2} x^2$$

Exemplo: $\int_0^3 \int_1^2 x^2 y \, dy \, dx$

$$\int_1^2 x^2 y \, dy = \left[x^2 \frac{y^2}{2} \right]_{y=1}^{y=2} = x^2 \left(\frac{2^2}{2} \right) - x^2 \left(\frac{1^2}{2} \right) = \tfrac{3}{2} x^2$$

$$\int_0^3 \int_1^2 x^2 y \, dy \, dx = \int_0^3 \left[\int_1^2 x^2 y \, dy \right] dx$$

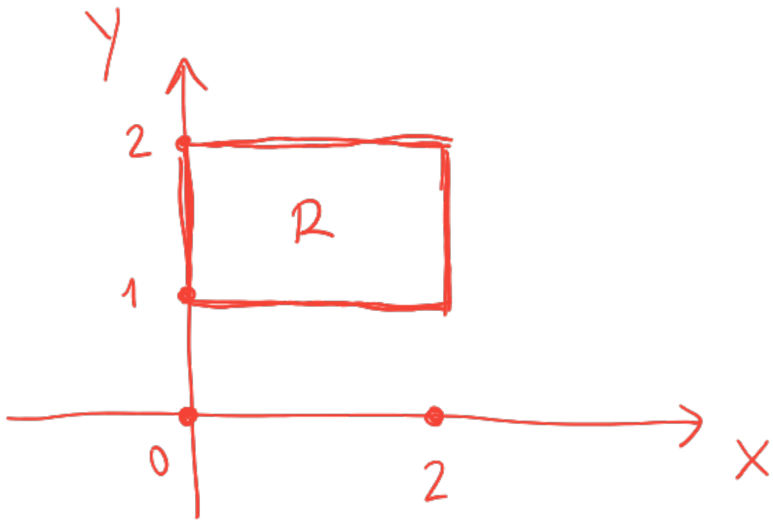
$$= \int_0^3 \tfrac{3}{2} x^2 \, dx = \frac{x^3}{2} \bigg]_0^3 = \frac{27}{2}$$

Teorema de Fubini: Se f é contínua em $R = \{(x, y) \mid a \leqslant x \leqslant b, c \leqslant y \leqslant d\}$, então

$$\iint_R f(x, y)\, dA = \int_a^b \int_c^d f(x, y)\, dy\, dx = \int_c^d \int_a^b f(x, y)\, dx\, dy$$

Exemplo: $\iint_R (x - 3y^2)\, dA,\ R = \{(x, y) \mid 0 \leqslant x \leqslant 2,\ 1 \leqslant y \leqslant 2\}$

$f(x,y) = x - 3y^2$



Exemplo: $\iint_R (x - 3y^2)\, dA$, $R = \{(x, y) \mid 0 \leqslant x \leqslant 2, 1 \leqslant y \leqslant 2\}$

$$\iint_R (x - 3y^2)\, dA = \int_0^2 \left[\int_1^2 (x - 3y^2)\, dy \right] dx = \int_0^2 \left[xy - y^3 \right]_{y=1}^{y=2} dx$$

$$= \int_0^2 (x - 7)\, dx = \frac{x^2}{2} - 7x \Big]_0^2 = -12$$

$$\int_1^2 x - 3y^2\, dy = xy - \frac{3y^3}{3} \Big|_{y=1}^{y=2} = (2x - 8) - (x - 1)$$

$$= x - 7$$

Exemplo: $\iint_R (x - 3y^2)\, dA,\ R = \{(x, y) \mid 0 \leqslant x \leqslant 2, 1 \leqslant y \leqslant 2\}$

$$\iint_R (x - 3y^2)\, dA = \int_0^2 \int_1^2 (x - 3y^2)\, dy\, dx = \int_0^2 \left[xy - y^3\right]_{y=1}^{y=2} dx$$

$$= \int_0^2 (x - 7)\, dx = \frac{x^2}{2} - 7x \Big]_0^2 = -12$$

$$\iint_R (x - 3y^2)\, dA = \int_1^2 \left[\int_0^2 (x - 3y^2)\, dx\right] dy$$

$$= \int_1^2 \left[\frac{x^2}{2} - 3xy^2\right]_{x=0}^{x=2} dy$$

$$= \int_1^2 (2 - 6y^2)\, dy = 2y - 2y^3\Big]_1^2 = -12$$

$$\int_0^2 x - 3y^2 dx = \frac{x^2}{2} - 3y^2 x \Big|_{x=0}^{x=2} = \frac{4}{2} - 6y^2 = 2 - 6y^2$$

Exemplo: $\iint_R y \sin(xy)\, dA,\ R = [1, 2] \times [0, \pi]$

$$\int u\,dv = u\cdot v - \int v\,du$$

Exemplo: $\iint_R y\sin(xy)\,dA,\ R = [1,2]\times[0,\pi]$

$$\iint_R y\sin(xy)\,dA = \int_1^2\left[\int_0^\pi y\sin(xy)\,dy\right]dx$$

$$\int y\cdot\sin(xy)\,dy = -\frac{y\cos(xy)}{x} - \int -\frac{\cos(xy)}{x}\,dy$$

$u = y \Rightarrow du = dy$

$dv = \sin(xy)\,dy \Rightarrow v = \int \sin(xy)\,dy = -\frac{\cos(xy)}{x}$

$$= -\frac{y\cos(xy)}{x} + \frac{1}{x}\int \cos(xy)\,dy$$

$$= -\frac{y\cos(xy)}{x} + \frac{\sin(xy)}{x^2}$$

$$\int \sin(xy)\,dy = \int \sin t\cdot\frac{1}{x}dt = \frac{1}{x}\int \sin t\,dt$$

$t = xy \Rightarrow dt = x\,dy$

$\Rightarrow dy = \frac{1}{x}dt$

$$= \frac{1}{x}(-\cos t) = \frac{1}{x}(-\cos(xy))$$

Exemplo: $\iint_R y \sin(xy)\, dA,\ R = [1, 2] \times [0, \pi]$

$$\iint_R y \sin(xy)\, dA = \int_1^2 \int_0^\pi y \sin(xy)\, dy\, dx$$

$$u = y \qquad dv = \sin(xy)\, dy$$

$$du = dy \qquad v = -\frac{\cos(xy)}{x}$$

Exemplo: $\iint_R y \sin(xy)\, dA,\ R = [1, 2] \times [0, \pi]$

$$\iint\limits_R y \sin(xy)\, dA = \int_1^2 \int_0^\pi y \sin(xy)\, dy\, dx$$

$$u = y \qquad dv = \sin(xy)\, dy$$

$$du = dy \qquad v = -\frac{\cos(xy)}{x}$$

$$\int_0^\pi y \sin(xy)\, dy = -\frac{y \cos(xy)}{x}\Bigg]_{y=0}^{y=\pi} + \frac{1}{x}\int_0^\pi \cos(xy)\, dy$$

$$= -\frac{\pi \cos \pi x}{x} + \frac{1}{x^2}\left[\sin(xy)\right]_{y=0}^{y=\pi}$$

$$= -\frac{\pi \cos \pi x}{x} + \frac{\sin \pi x}{x^2}$$

Exemplo: $\iint_R y \sin(xy)\, dA,\ R = [1, 2] \times [0, \pi]$

$$\iint_R y \sin(xy)\, dA = \int_1^2 \int_0^\pi y \sin(xy)\, dy\, dx$$

$$u = y \qquad dv = \sin(xy)\, dy$$

$$du = dy \qquad v = -\frac{\cos(xy)}{x}$$

$$\int_0^\pi y \sin(xy)\, dy = -\frac{y \cos(xy)}{x}\Bigg]_{y=0}^{y=\pi} + \frac{1}{x}\int_0^\pi \cos(xy)\, dy$$

$$= -\frac{\pi \cos \pi x}{x} + \frac{1}{x^2}\left[\sin(xy)\right]_{y=0}^{y=\pi}$$

$$= -\frac{\pi \cos \pi x}{x} + \frac{\sin \pi x}{x^2}$$

$$\int \left(-\frac{\pi \cos \pi x}{x}\right) dx = -\frac{\sin \pi x}{x} - \int \frac{\sin \pi x}{x^2}\, dx$$

$$u = -1/x \qquad dv = \pi \cos \pi x\, dx$$

$$du = dx/x^2 \qquad v = \sin \pi x$$

$$\int \left(-\frac{\pi \cos \pi x}{x} \right) dx = -\frac{\sin \pi x}{x} - \int \frac{\sin \pi x}{x^2} \, dx$$

$$\int \left(-\frac{\pi \cos \pi x}{x} + \frac{\sin \pi x}{x^2} \right) dx = -\frac{\sin \pi x}{x}$$

$$\int \left(-\frac{\pi \cos \pi x}{x}\right) dx = -\frac{\sin \pi x}{x} - \int \frac{\sin \pi x}{x^2}\, dx$$

$$\int \left(-\frac{\pi \cos \pi x}{x} + \frac{\sin \pi x}{x^2}\right) dx = -\frac{\sin \pi x}{x}$$

$$\int_1^2 \int_0^\pi y \sin(xy)\, dy\, dx = \left[-\frac{\sin \pi x}{x}\right]_1^2$$

$$= -\frac{\sin 2\pi}{2} + \sin \pi = 0$$

Solução alternativa:

Solução alternativa:

$$\iint_R y\sin(xy)\,dA = \int_0^{\pi}\left[\int_1^2 y\sin(xy)\,dx\right]dy = \int_0^{\pi}\Big[-\cos(xy)\Big]_{x=1}^{x=2}\,dy$$

$$= \int_0^{\pi}(-\cos 2y + \cos y)\,dy$$

$$= -\tfrac{1}{2}\sin 2y + \sin y\Big]_0^{\pi} = 0$$

$$\int_1^2 y\sin(xy)\,dx = y\int_1^2 \sin(xy)\,dx = y\left(-\frac{\cos(xy)}{y}\Bigg|_{x=1}^{x=2}\right)$$

Suponha $f(x, y) = g(x)h(y)$

$$f(x,y) = \underbrace{\cos(x)}\cdot\underbrace{y^2}$$

$$\underbrace{x^2}\cdot\underbrace{y}$$

$$\underbrace{\ln x}\cdot\underbrace{\cos y}$$

$$\underbrace{\frac{x^2-2x}{\sqrt{x}}}\cdot\underbrace{e^y}$$

Suponha $f(x, y) = g(x)h(y)$

$$\iint_R f(x, y)\, dA = \int_c^d \int_a^b g(x)h(y)\, dx\, dy = \int_c^d \left[\int_a^b g(x)h(y)\, dx \right] dy$$

Suponha $f(x, y) = g(x)h(y)$

$$\iint_R f(x, y)\, dA = \int_c^d \int_a^b g(x)h(y)\, dx\, dy = \int_c^d \left[\int_a^b g(x)h(y)\, dx \right] dy$$

$$= \int_c^d \left[h(y) \left(\int_a^b g(x)\, dx \right) \right] dy = \int_a^b g(x)\, dx \int_c^d h(y)\, dy$$

Suponha $f(x, y) = g(x)h(y)$

$$\iint_R f(x, y)\, dA = \int_c^d \int_a^b g(x)h(y)\, dx\, dy = \int_c^d \left[\int_a^b g(x)h(y)\, dx \right] dy$$

$$= \int_c^d \left[h(y) \left(\int_a^b g(x)\, dx \right) \right] dy = \int_a^b g(x)\, dx \int_c^d h(y)\, dy$$

$$\iint_R g(x)h(y)\, dA = \int_a^b g(x)\, dx \int_c^d h(y)\, dy \qquad R = [a, b] \times [c, d]$$

Exemplo: $R = [0, \pi/2] \times [0, \pi/2]$

$$\iint_R \sin x \, \cos y \, dA = \int_0^{\pi/2} \sin x \, dx \int_0^{\pi/2} \cos y \, dy$$

$$= \left[-\cos x\right]_0^{\pi/2} \left[\sin y\right]_0^{\pi/2} = 1 \cdot 1 = 1$$