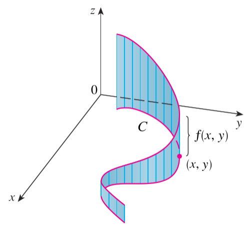
Cálculo III

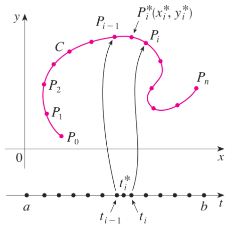
Integral de linha

Prof. Adriano Barbosa

Integral de linha



Integral de linha



C dada pelas equações paramétricas

$$x = x(t) \qquad y = y(t) \qquad a \leqslant t \leqslant b$$

$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}$$

$$\sum_{i=1}^{n} f(x_i^*, y_i^*)\,\Delta s_i$$

Integral de linha

Se f é definida sobre uma curva suave C, então a **integral de linha de f sobre C** é

$$\int_C f(x, y)\,ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\,\Delta s_i$$

se esse limite existir.

Integral de linha

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$$\int_C f(x, y)\,ds = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

Exemplo

Calcule $\int_C (2 + x^2y)\,ds$, onde C é a metade superior do círculo unitário $x^2 + y^2 = 1$.

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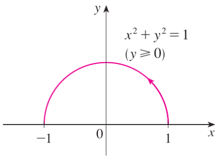

$x = \cos t$
$y = \operatorname{sen} t$
$0 \leqslant t \leqslant \pi$

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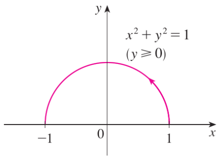

$x = \cos t$
$y = \operatorname{sen} t$
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$$\int_C (2 + x^2y)\, ds = \int_0^\pi (2 + \cos^2 t \operatorname{sen} t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

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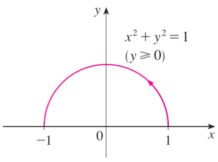

$x = \cos t$
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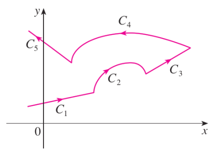
$$\int_C (2 + x^2y)\, ds = \int_0^\pi (2 + \cos^2 t \operatorname{sen} t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^\pi (2 + \cos^2 t \operatorname{sen} t)\sqrt{\operatorname{sen}^2 t + \cos^2 t}\, dt$$

$$= \int_0^\pi (2 + \cos^2 t \operatorname{sen} t)\, dt = \left[2t - \frac{\cos^3 t}{3}\right]_0^\pi$$

$$= 2\pi + \tfrac{2}{3}$$

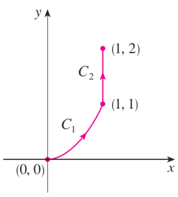
Integração sobre curvas suaves por partes



$$\int_C f(x, y)\, ds = \int_{C_1} f(x, y)\, ds + \int_{C_2} f(x, y)\, ds + \cdots + \int_{C_n} f(x, y)\, ds$$

Exemplo

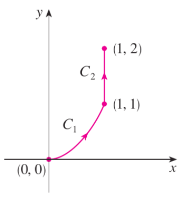
Calcule $\int_C 2x\, ds$, onde C é formada pelo arco C_1 da parábola $y = x^2$ de $(0, 0)$ a $(1, 1)$ seguido pelo segmento de reta vertical C_2 de $(1, 1)$ a $(1, 2)$.



$$\int_C 2x\, ds = \int_{C_1} 2x\, ds + \int_{C_2} 2x\, ds$$

Exemplo

Calcule $\int_C 2x\, ds$, onde C é formada pelo arco C_1 da parábola $y = x^2$ de $(0, 0)$ a $(1, 1)$ seguido pelo segmento de reta vertical C_2 de $(1, 1)$ a $(1, 2)$.

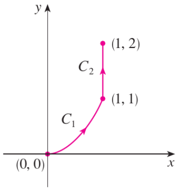


$x = x \qquad y = x^2 \qquad 0 \leqslant x \leqslant 1$

$$\int_{C_1} 2x\, ds = \int_0^1 2x\sqrt{\left(\frac{dx}{dx}\right)^2 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Exemplo

Calcule $\int_C 2x\, ds$, onde C é formada pelo arco C_1 da parábola $y = x^2$ de (0, 0) a (1, 1) seguido pelo segmento de reta vertical C_2 de (1, 1) a (1, 2).



$$x = x \qquad y = x^2 \qquad 0 \leqslant x \leqslant 1$$

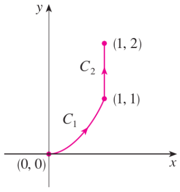
$$\int_{C_1} 2x\, ds = \int_0^1 2x \sqrt{\left(\frac{dx}{dx}\right)^2 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_0^1 2x\sqrt{1 + 4x^2}\, dx$$

$$= \tfrac{1}{4} \cdot \tfrac{2}{3}(1 + 4x^2)^{3/2}\Big]_0^1 = \frac{5\sqrt{5} - 1}{6}$$

Exemplo

Calcule $\int_C 2x\, ds$, onde C é formada pelo arco C_1 da parábola $y = x^2$ de (0, 0) a (1, 1) seguido pelo segmento de reta vertical C_2 de (1, 1) a (1, 2).

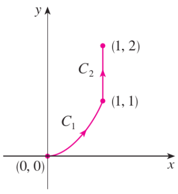


$$x = 1 \qquad y = y \qquad 1 \leqslant y \leqslant 2$$

$$\int_{C_2} 2x\, ds = \int_1^2 2(1) \sqrt{\left(\frac{dx}{dy}\right)^2 + \left(\frac{dy}{dy}\right)^2}\, dy$$

Exemplo

Calcule $\int_C 2x\, ds$, onde C é formada pelo arco C_1 da parábola $y = x^2$ de (0, 0) a (1, 1) seguido pelo segmento de reta vertical C_2 de (1, 1) a (1, 2).



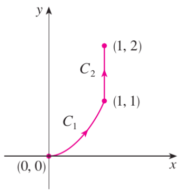
$$x = 1 \qquad y = y \qquad 1 \leqslant y \leqslant 2$$

$$\int_{C_2} 2x\, ds = \int_1^2 2(1) \sqrt{\left(\frac{dx}{dy}\right)^2 + \left(\frac{dy}{dy}\right)^2}\, dy$$

$$= \int_1^2 2\, dy = 2$$

Exemplo

Calcule $\int_C 2x\, ds$, onde C é formada pelo arco C_1 da parábola $y = x^2$ de (0, 0) a (1, 1) seguido pelo segmento de reta vertical C_2 de (1, 1) a (1, 2).



$$\int_C 2x\, ds = \int_{C_1} 2x\, ds + \int_{C_2} 2x\, ds$$

$$= \frac{5\sqrt{5} - 1}{6} + 2$$

Integral de linha

$x = x(t), \;\; y = y(t), \;\; dx = x'(t)\; dt, \;\; dy = y'(t)\, dt$

$$\int_C f(x, y)\, dx = \int_a^b f\big(x(t), y(t)\big)\, x'(t)\, dt$$

$$\int_C f(x, y)\, dy = \int_a^b f\big(x(t), y(t)\big)\, y'(t)\, dt$$

Integral de linha

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Notação

$$\int_C P(x, y)\, dx + \int_C Q(x, y)\, dy = \int_C P(x, y)\, dx + Q(x, y)\, dy$$

Integral de linha

$$\int_{-C} f(x, y)\, dx = -\int_C f(x, y)\, dx$$

$$\int_{-C} f(x, y)\, dy = -\int_C f(x, y)\, dy$$

$$\int_{-C} f(x, y)\, ds = \int_C f(x, y)\, ds$$

Integral de linha no espaço

$$\int_C f(x, y, z)\, ds = \int_a^b f(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

$$\int_C P(x, y, z)\, dx + Q(x, y, z)\, dy + R(x, y, z)\, dz$$

De modo geral

$$\int_a^b f(\mathbf{r}(t))\, |\mathbf{r}'(t)|\, dt$$

Exercício

Calcule $\int_C y^2\, dx + x\, dy$, onde

(a) $C = C_1$ é o segmento de reta de $(-5, -3)$ a $(0, 2)$

(b) $C = C_2$ é o arco da parábola $x = 4 - y^2$ de $(-5, -3)$ a $(0, 2)$