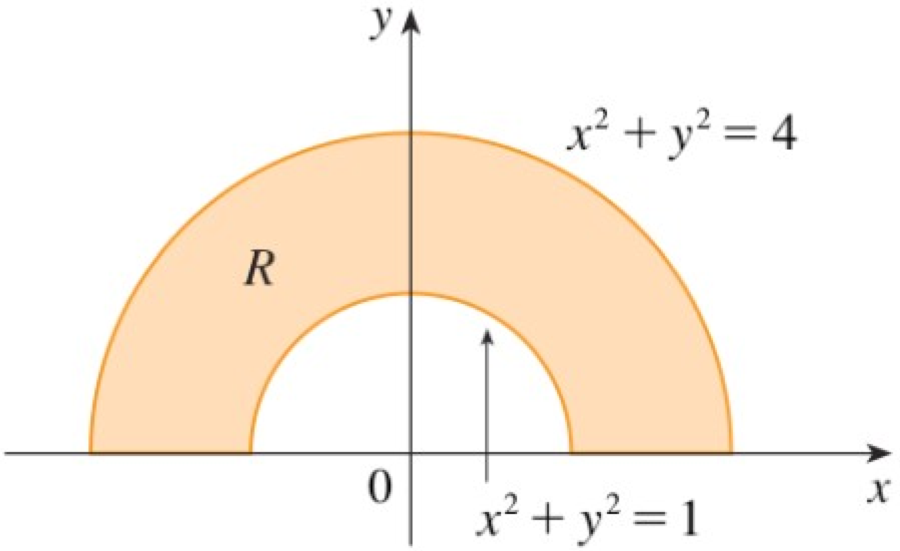
Cálculo III

Coordenadas polares

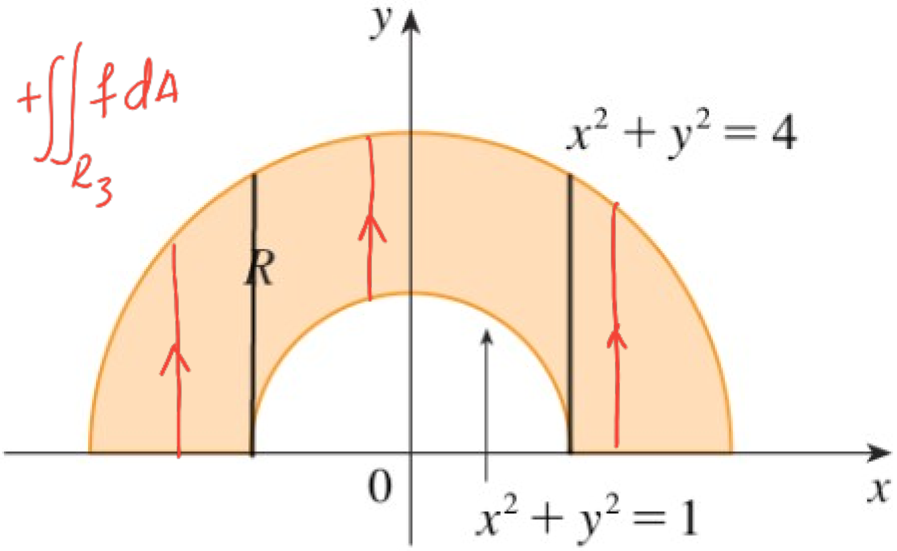
Prof. Adriano Barbosa

Coordenadas polares



Coordenadas polares

$$\iint_R f\,dA = \iint_{R_1} f\,dA + \iint_{R_2} f\,dA + \iint_{R_3} f\,dA$$



$$R_1 = \left\{(x, y); -2 \leq x \leq -1 \text{ e } 0 \leq y \leq \sqrt{4 - x^2}\right\}$$

$$R_2 = \left\{(x, y); -1 \leq x \leq 1 \text{ e } \sqrt{1 - x^2} \leq y \leq \sqrt{4 - x^2}\right\}$$

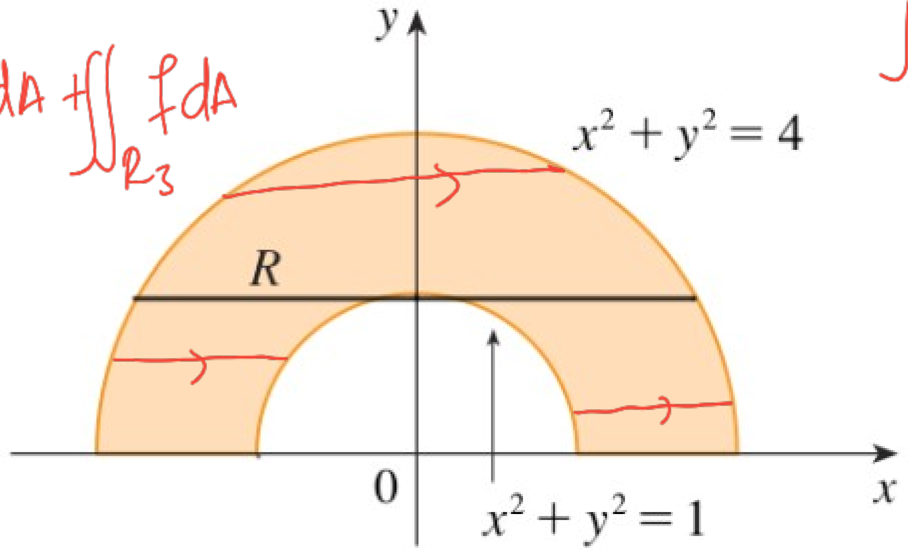
$$R_3 = \left\{(x, y); 1 \leq x \leq 2 \text{ e } 0 \leq y \leq \sqrt{4 - x^2}\right\}$$

Coordenadas polares

$$\iint_R f\,dA = \iint_{R_1} f\,dA + \iint_{R_2} f\,dA + \iint_{R_3} f\,dA$$

$$\iint_{R_1} f\,dA = \int_0^1 \int_{-\sqrt{4-y^2}}^{-\sqrt{1-y^2}} f(x,y)\,dx\,dy$$

Novamente!

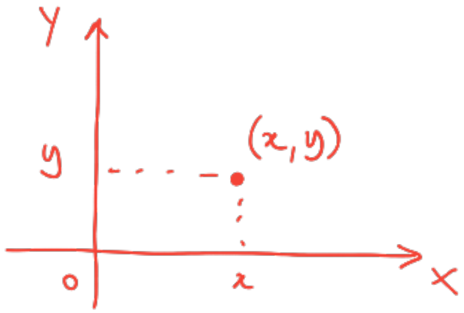


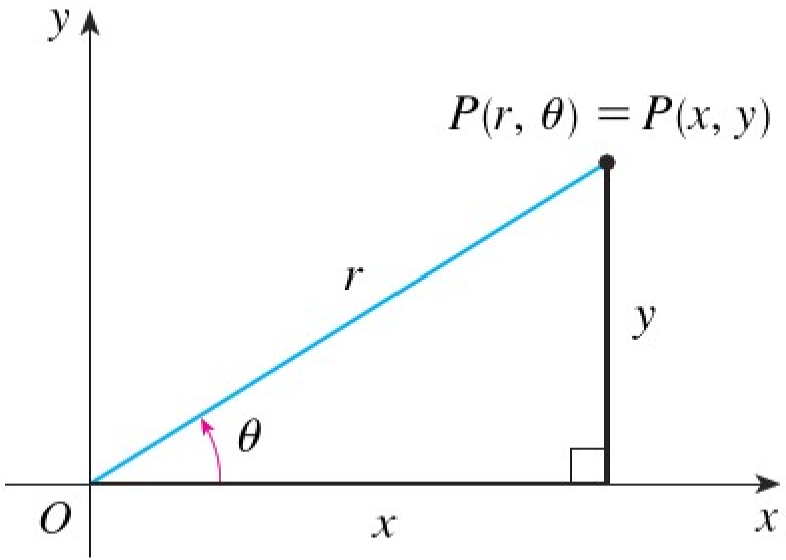
$$R_1 = \left\{(x,y); -\sqrt{4-y^2} \le x \le -\sqrt{1-y^2} \text{ e } 0 \le y \le 1\right\}$$

$$R_2 = \left\{(x,y); -\sqrt{4-y^2} \le x \le \sqrt{4-y^2} \text{ e } 1 \le y \le 2\right\}$$

$$R_3 = \left\{(x,y); \sqrt{1-y^2} \le x \le \sqrt{4-y^2} \text{ e } 0 \le y \le 1\right\}$$

Coordenadas polares

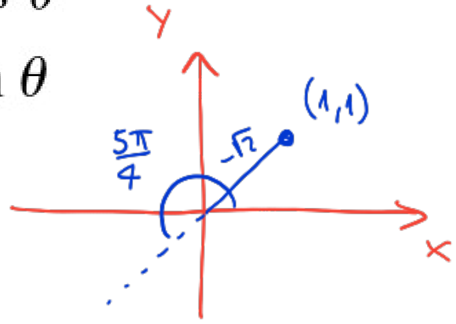




$$r^2 = x^2 + y^2$$

$$x = r\cos\theta$$

$$y = r\operatorname{sen}\theta$$



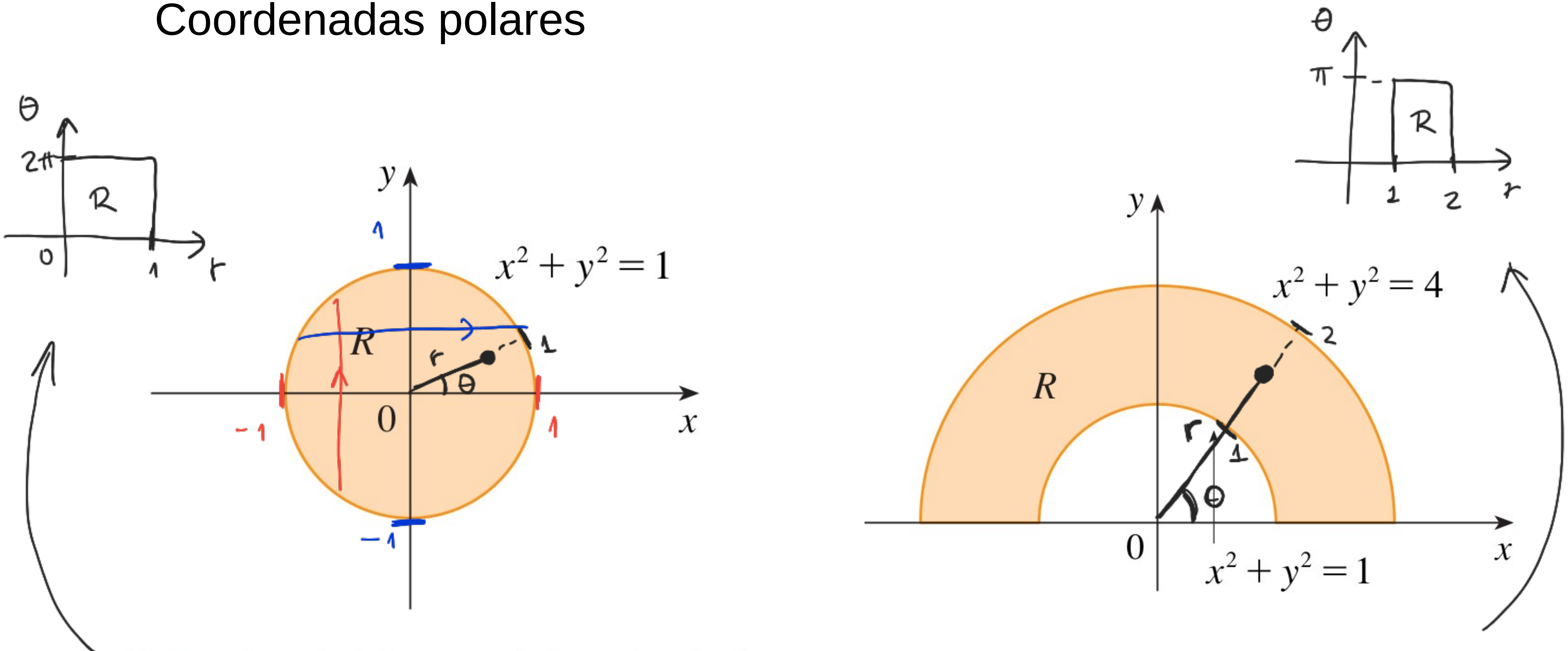
$r^2 = 1^2 + 1^2 = 2 \Rightarrow r = \sqrt{2}$

$\cos\theta = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$, $\operatorname{sen}\theta = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \Rightarrow \theta = \frac{\pi}{4}$

$(1,1)_c = \left(\sqrt{2}, \frac{\pi}{4}\right)_p = \left(-\sqrt{2}, \frac{5\pi}{4}\right)_p$

Não é único!

Coordenadas polares



(a) $R = \{(r, \theta) \mid 0 \leqslant r \leqslant 1, 0 \leqslant \theta \leqslant 2\pi\}$

(b) $R = \{(r, \theta) \mid 1 \leqslant r \leqslant 2, 0 \leqslant \theta \leqslant \pi\}$

$R = \{ (x,y) \mid -1 \leq x \leq 1 \,,\, -\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2} \}$

$R = \{ (x,y) \mid -1 \leq y \leq 1 \,,\, -\sqrt{1-y^2} \leq x \leq \sqrt{1-y^2} \}$

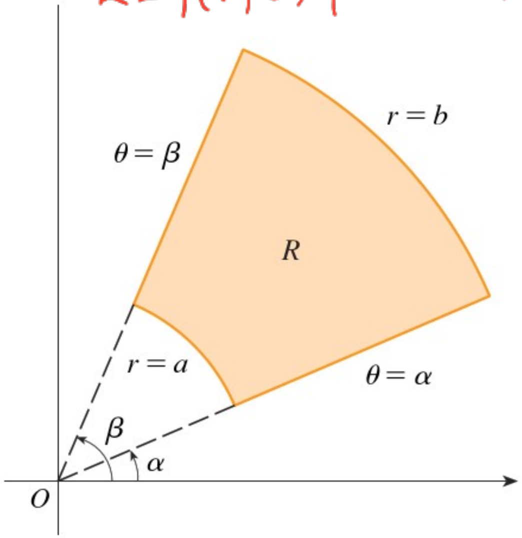
Integração sobre regiões circulares

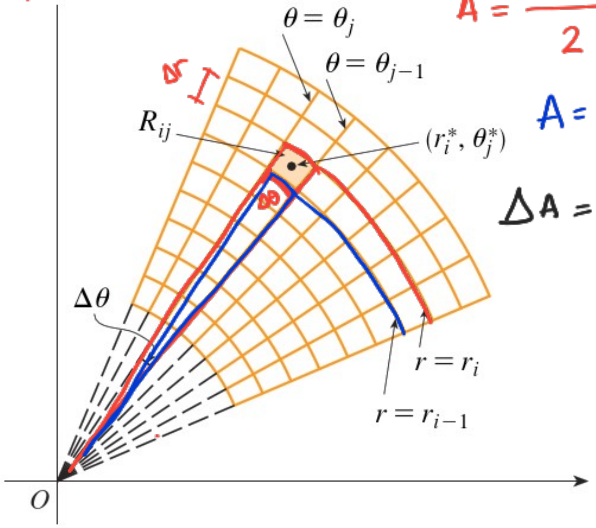
$A = \pi r^2$

$$\frac{\cancel{\pi} r^2}{2\cancel{\pi}} = \frac{A}{\theta}$$

$$\Rightarrow A = \frac{\theta r^2}{2}$$

$R = \{(r,\theta) \mid a \le r \le b, \alpha \le \theta \le \beta\}$





$$A = \frac{\Delta\theta \cdot r_i^2}{2}$$

$$A = \frac{\Delta\theta \cdot r_{i-1}^2}{2}$$

$$\Delta A = A - A$$

$$R_{ij} = \{(r,\theta) \mid r_{i-1} \leqslant r \leqslant r_i,\ \theta_{j-1} \leqslant \theta \leqslant \theta_j\}$$

$$r_i^* = \tfrac{1}{2}(r_{i-1} + r_i) \qquad \theta_j^* = \tfrac{1}{2}(\theta_{j-1} + \theta_j)$$

$$\Delta A_i = \tfrac{1}{2} r_i^2 \Delta\theta - \tfrac{1}{2} r_{i-1}^2 \Delta\theta = \tfrac{1}{2}(r_i^2 - r_{i-1}^2)\Delta\theta$$

$$= \tfrac{1}{2}(r_i + r_{i-1})(r_i - r_{i-1})\Delta\theta = r_i^* \Delta r \Delta\theta$$

$\Delta r = r_i - r_{i-1}$ $\qquad$ $\Delta\theta = \theta_j - \theta_{j-1}$ $\qquad$ $V \approx \sum_i \sum_j f(r_i^*, \theta_j^*) \Delta A_i$

Integração sobre regiões circulares

Se f é contínua

R dado por $0 \leqslant a \leqslant r \leqslant b$, $\alpha \leqslant \theta \leqslant \beta$, onde $0 \leqslant \beta - \alpha \leqslant 2\pi$, então

$$\iint_R f(x, y)\, dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\,\mathrm{sen}\,\theta)\, r\, dr\, d\theta$$

Integração sobre regiões circulares

- Trocamos a variação do x e do y pela variação do raio e do ângulo;
- Trocamos na regra da função:

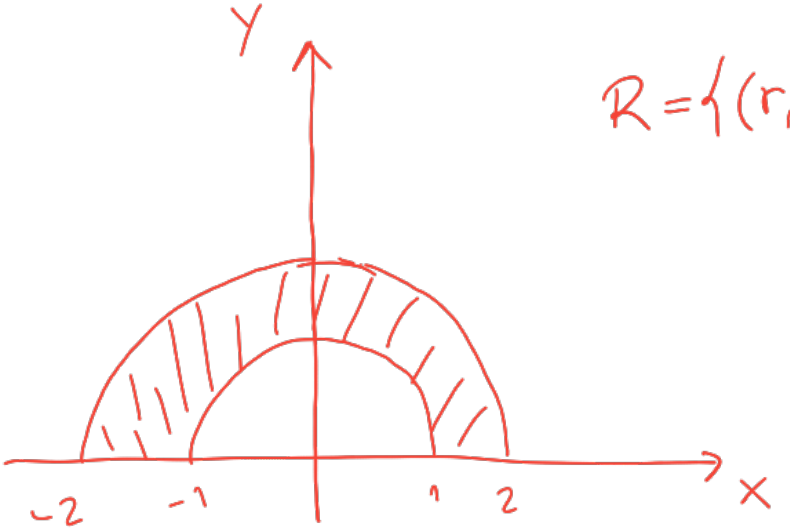
$$x \text{ por } r\cos\theta \text{ e } y \text{ por } r\text{sen}\theta.$$

- Trocamos a variação dos retângulos cartesianos $dA = dx\,dy$ pela variação dos retângulos polares $r\,dr\,d\theta$.

Exemplo

Calcule $\iint_R (3x + 4y^2)\, dA$, onde R é a região no semiplano superior limitada pelos círculos $x^2 + y^2 = 1$ e $x^2 + y^2 = 4$.

$$R = \{(r,\theta) \mid 0 \leq \theta \leq \pi,\ 1 \leq r \leq 2\}$$



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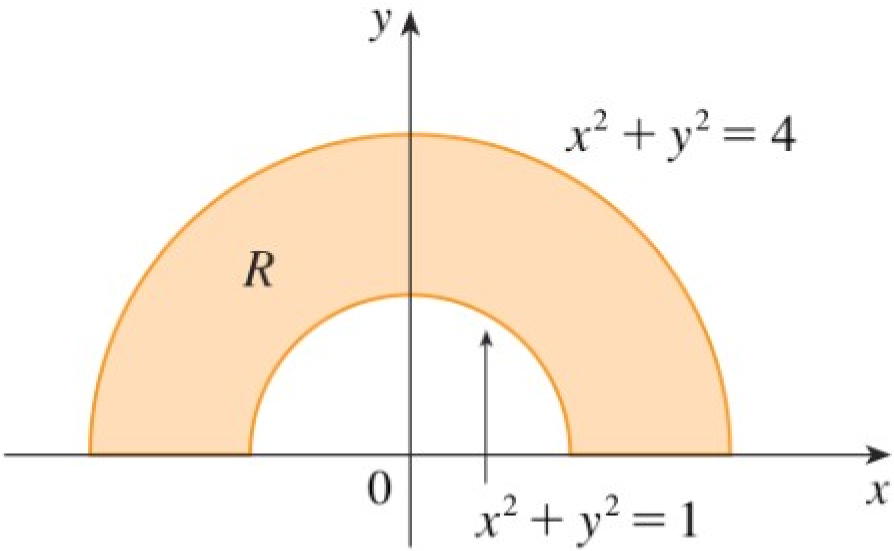
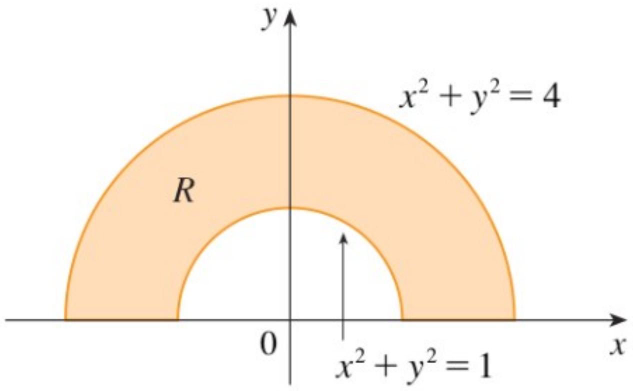


Exemplo

$$\iint_R (3x + 4y^2)\, dA = \int_0^{\pi} \int_1^2 (3r\cos\theta + 4r^2 \operatorname{sen}^2\theta)\, r\, dr\, d\theta$$



$$R = \{(r, \theta) \mid 1 \leqslant r \leqslant 2,\ 0 \leqslant \theta \leqslant \pi\}$$

$$\operatorname{sen}^2\theta = \frac{1 - \cos(2\theta)}{2} \qquad \cos^2\theta = \frac{1 + \cos(2\theta)}{2}$$

$$\operatorname{sen}^2\theta + \cos^2\theta = 1 \Rightarrow \cos^2\theta = 1 - \operatorname{sen}^2\theta$$

$$\cos(\alpha + \beta) = \cos\alpha \cos\beta - \operatorname{sen}\alpha \operatorname{sen}\beta$$

$$\cos(2\theta) = \cos^2\theta - \operatorname{sen}^2\theta$$

$$= 1 - \operatorname{sen}^2\theta - \operatorname{sen}^2\theta$$

$$= 1 - 2\operatorname{sen}^2\theta$$

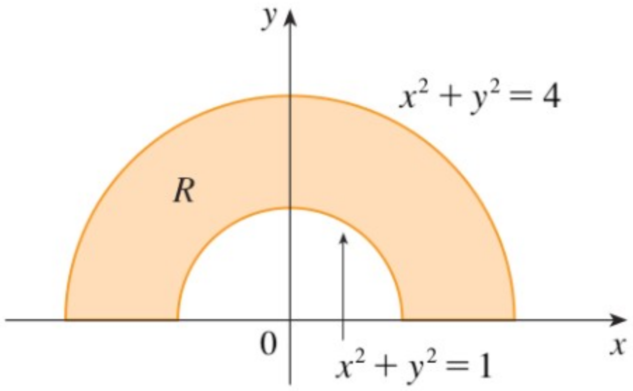
$$\Rightarrow \operatorname{sen}^2\theta = \frac{1 - \cos(2\theta)}{2}$$

$$\int \cos 2\theta \, d\theta = \int \cos u \cdot \frac{1}{2} du = \frac{1}{2}\int \cos u \, du = \frac{1}{2} \operatorname{sen} u + c = \frac{1}{2} \operatorname{sen} 2\theta + c$$

Exemplo

$u = 2\theta$

$du = 2d\theta$



$$R = \{(r, \theta) \mid 1 \leqslant r \leqslant 2, 0 \leqslant \theta \leqslant \pi\}$$

$$\iint_R (3x + 4y^2)\, dA = \int_0^\pi \int_1^2 (3r\cos\theta + 4r^2 \operatorname{sen}^2\theta)\, r\, dr\, d\theta$$

$$= \int_0^\pi \left[\int_1^2 (3r^2\cos\theta + 4r^3 \operatorname{sen}^2\theta)\, dr\right] d\theta$$

$$= \int_0^\pi \left[r^3\cos\theta + r^4 \operatorname{sen}^2\theta\right]_{r=1}^{r=2} d\theta$$

$$= \int_0^\pi (7\cos\theta + 15 \operatorname{sen}^2\theta)\, d\theta$$

$$= \int_0^\pi \left[7\cos\theta + \tfrac{15}{2}(1 - \cos 2\theta)\right] d\theta$$

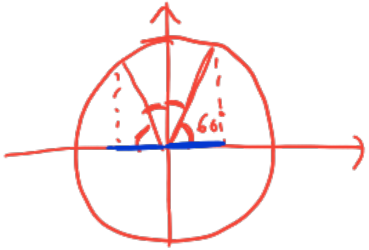
$$7\int \cos\theta \, d\theta + 15\int \operatorname{sen}^2\theta \, d\theta$$

$$7\int \cos\theta \, d\theta + 15\int \frac{1 - \cos 2\theta}{2} d\theta$$

$$= 7\operatorname{sen}\theta + \frac{15\theta}{2} - \frac{15}{4}\operatorname{sen} 2\theta \Big]_0^\pi = \frac{15\pi}{2}$$

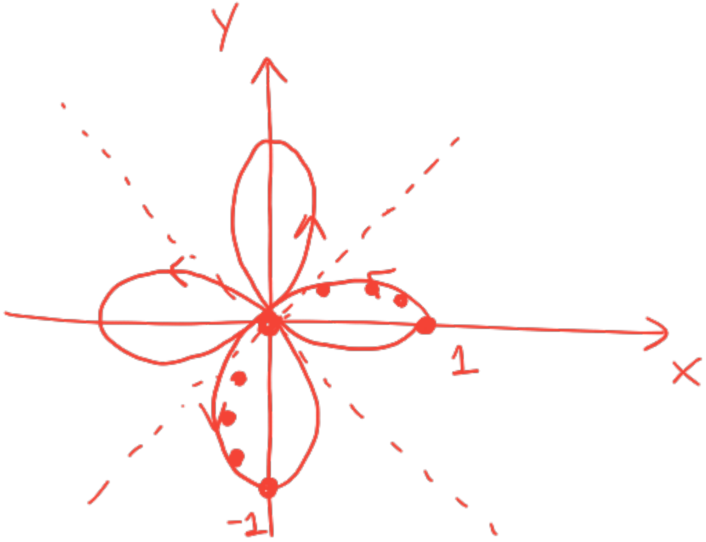
Exemplo

	30	45	60
s	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$
c	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$



$120^\circ = 90^\circ + 30^\circ$

Use a integral dupla para determinar a área contida em um laço da rosácea de quatro pétalas $r = \cos 2\theta$.

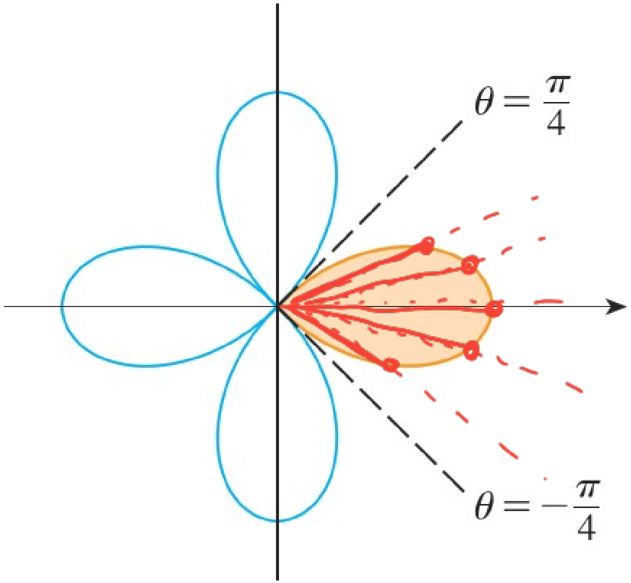


θ	r
0	1
15°	$\approx 0{,}86$
$22{,}5^\circ$	$\approx 0{,}7$
30°	$0{,}5$
45°	0
60°	$-0{,}5$
$67{,}5^\circ$	$\approx -0{,}7$
75°	$\approx -0{,}86$

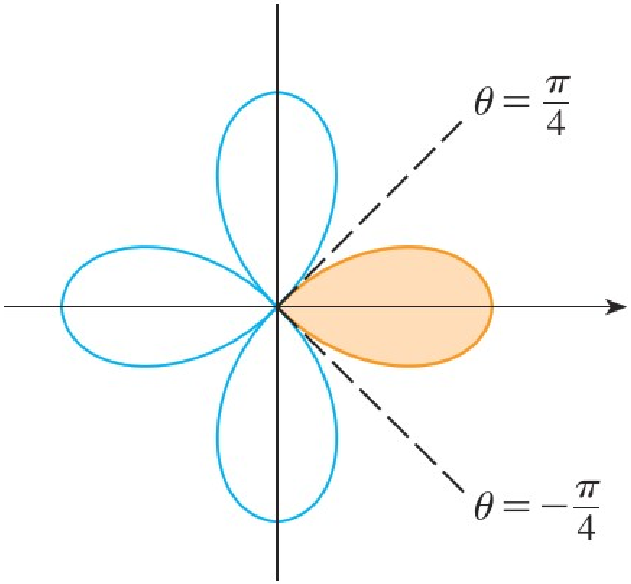
θ	r
90°	-1
$\vdots$	$\vdots$

Exemplo

Use a integral dupla para determinar a área contida em um laço da rosácea de quatro pétalas $r = \cos 2\theta$.

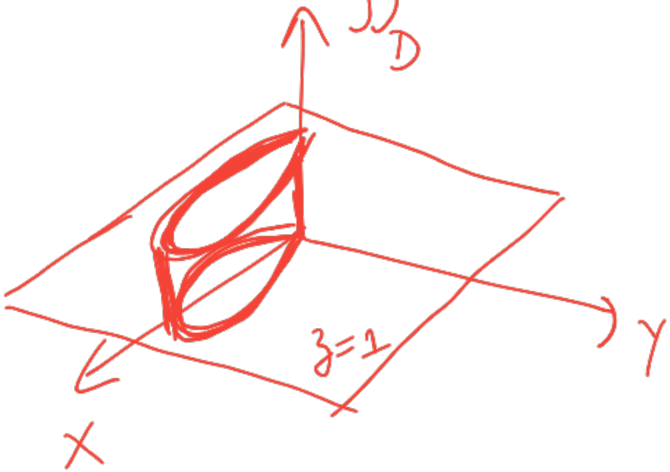


Exemplo

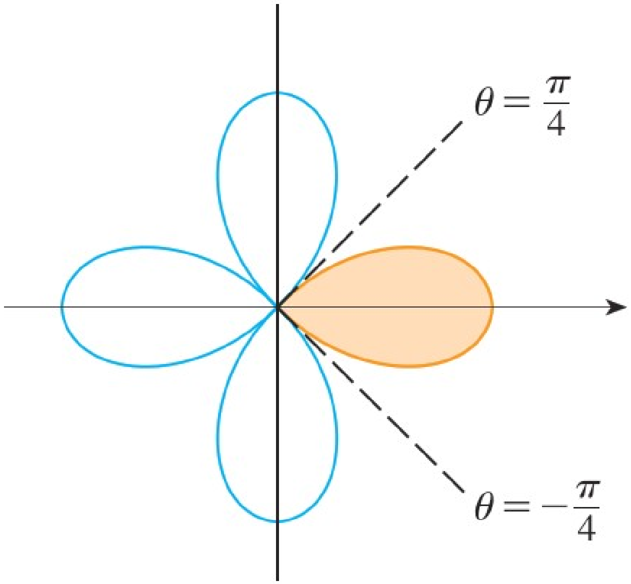


$$D = \{(r, \theta) \mid -\pi/4 \leqslant \theta \leqslant \pi/4,\ 0 \leqslant r \leqslant \cos 2\theta\}$$

$$\iint_D dA = V = A_b \cdot h = A_D \cdot 1 = A_D$$



Exemplo



$$D = \{(r, \theta) \mid -\pi/4 \leqslant \theta \leqslant \pi/4,\ 0 \leqslant r \leqslant \cos 2\theta\}$$

$$A(D) = \iint_D dA = \int_{-\pi/4}^{\pi/4} \int_0^{\cos 2\theta} r\, dr\, d\theta$$

$$= \int_{-\pi/4}^{\pi/4} \left[\tfrac{1}{2} r^2\right]_0^{\cos 2\theta} d\theta = \tfrac{1}{2} \int_{-\pi/4}^{\pi/4} \cos^2 2\theta\, d\theta$$

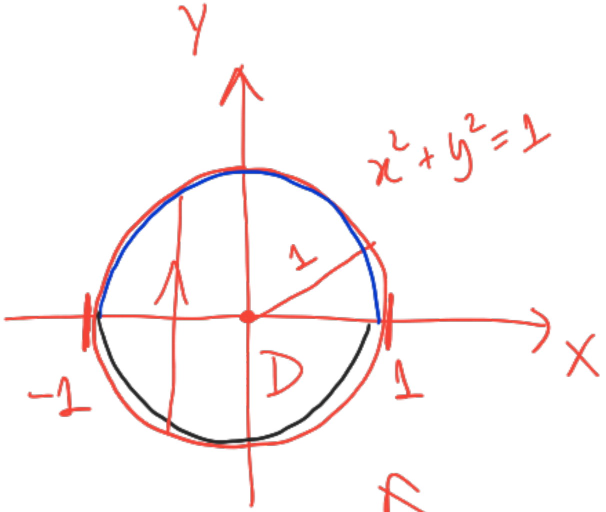
$$= \tfrac{1}{4} \int_{-\pi/4}^{\pi/4} (1 + \cos 4\theta)\, d\theta = \tfrac{1}{4}\left[\theta + \tfrac{1}{4} \operatorname{sen} 4\theta\right]_{-\pi/4}^{\pi/4} = \frac{\pi}{8}$$

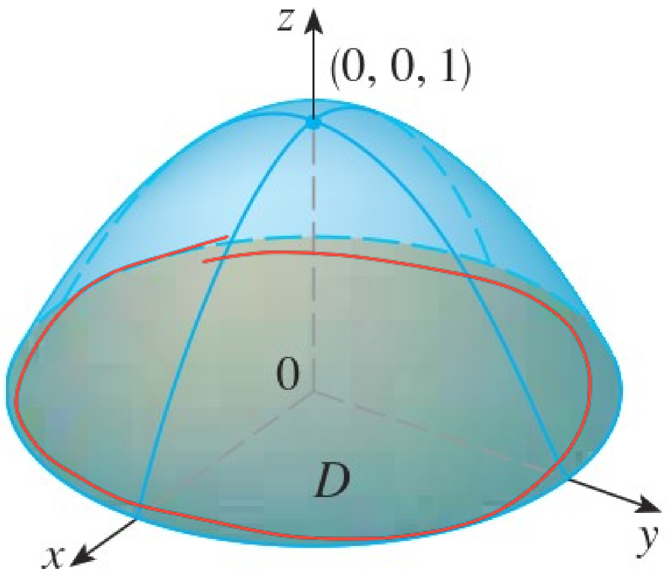
Exemplo

Determine o volume do sólido limitado pelo plano $z = 0$ e pelo paraboloide $z = 1 - x^2 - y^2$.

Exemplo

Determine o volume do sólido limitado pelo plano $z = 0$ e pelo paraboloide $z = 1 - x^2 - y^2$.





$$\begin{cases} z = 0 \\ z = 1 - x^2 - y^2 \end{cases}$$

$$\Downarrow$$

$$x^2 + y^2 = 1$$

$$V = \iint_D 1 - x^2 - y^2 \, dA$$

$$D = \{(x,y) \mid -1 \le x \le 1, -\sqrt{1-x^2} \le y \le \sqrt{1-x^2}\}$$

Exemplo

Se trabalhássemos com coordenadas retangulares

$$V = \iint_D (1 - x^2 - y^2)\, dA = \int_{-1}^{1} \left[\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (1 - x^2 - y^2)\, dy \right] dx$$

$$= \int_{-1}^{1} \left(y - x^2 y - \frac{y^3}{3} \Big|_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}} \right) dx$$

$$= \int_{-1}^{1} \sqrt{1-x^2} - x^2\sqrt{1-x^2} - \frac{\left(\sqrt{1-x^2}\right)^3}{3} - \left[-\sqrt{1-x^2} + x^2\sqrt{1-x^2} + \frac{\left(\sqrt{1-x^2}\right)^3}{3} \right] dx$$

Exemplo

Se trabalhássemos com coordenadas retangulares

$$V = \iint_D (1 - x^2 - y^2)\, dA = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (1 - x^2 - y^2)\, dy\, dx$$

Em coordenadas polares, D é dado por $0 \leqslant r \leqslant 1, 0 \leqslant \theta \leqslant 2\pi$.

$$V = \iint_D (1 - x^2 - y^2)\, dA = \int_0^{2\pi} \int_0^1 (1 - r^2)\, r\, dr\, d\theta$$

$$1 - r^2\cos^2\theta - r^2 \operatorname{sen}^2\theta$$

$$= 1 - r^2(\cos^2\theta + \operatorname{sen}^2\theta)$$

$$= 1 - r^2$$

Exemplo

Se trabalhássemos com coordenadas retangulares

$$V = \iint_D (1 - x^2 - y^2)\, dA = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (1 - x^2 - y^2)\, dy\, dx$$

Em coordenadas polares, D é dado por $0 \leqslant r \leqslant 1, 0 \leqslant \theta \leqslant 2\pi$.

não depende de θ

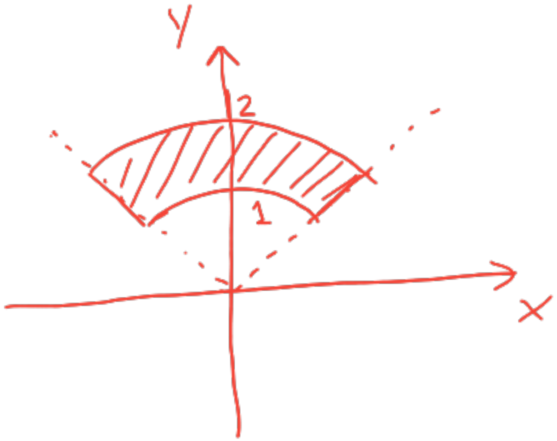
$$V = \iint_D (1 - x^2 - y^2)\, dA = \int_0^{2\pi} \left[\int_0^1 (1 - r^2)\, r\, dr \right] d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^1 (r - r^3)\, dr = 2\pi \left[\frac{r^2}{2} - \frac{r^4}{4} \right]_0^1 = \frac{\pi}{2}$$

Exercícios

Esboce a região cuja área é dada pela integral e calcule-a.

$$\int_{\pi/4}^{3\pi/4} \int_1^2 r \, dr \, d\theta$$



$3\pi/4$

$R = \{(r,\theta) \mid 1 \le r \le 2,\ \pi/4 \le \theta \le 3\pi/4\}$

$f(x,y) = 1$

Calcule a integral dada, colocando-a em coordenadas polares.
$\iint_D x^2 y \, dA$, onde D é a metade superior do disco com centro na origem e raio 5

$\frac{1250}{3}$