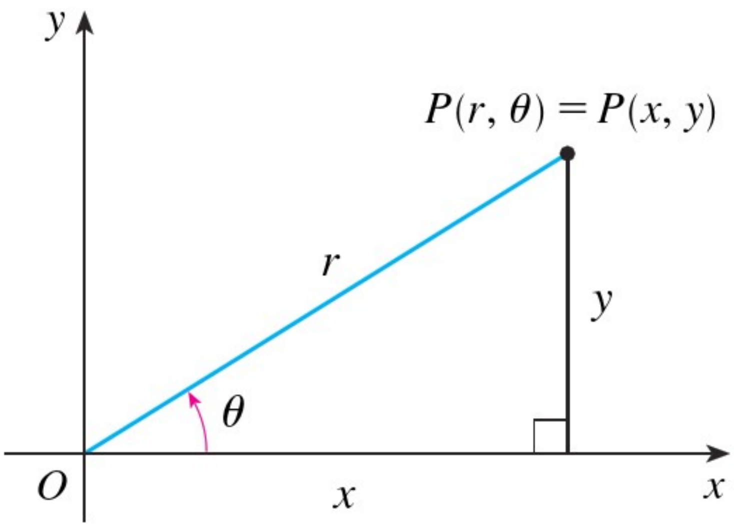
Cálculo III

Coordenadas cilíndricas e esféricas

Prof. Adriano Barbosa

Coordenadas polares



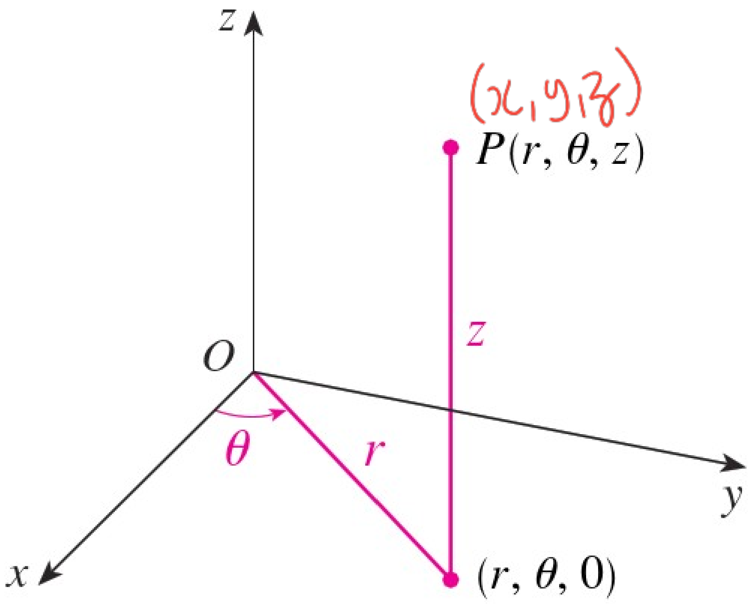
$$x = r \cos \theta$$

$$y = r \operatorname{sen} \theta$$

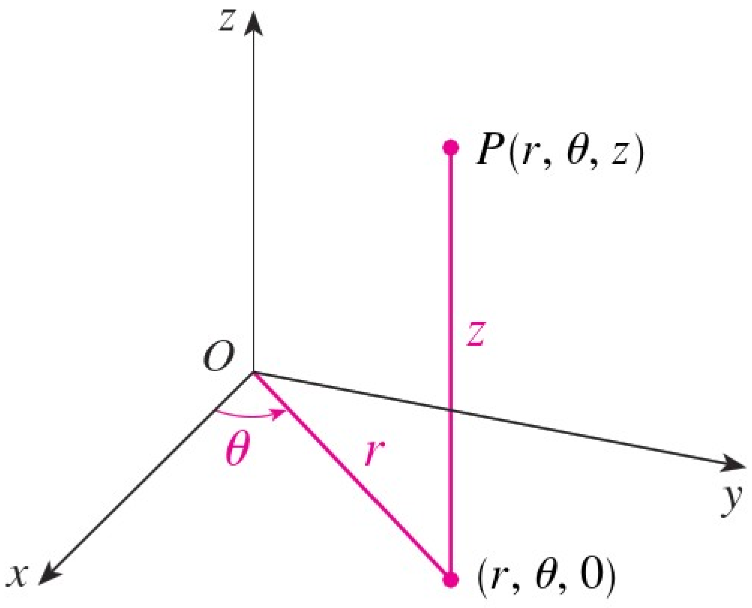
$$r^2 = x^2 + y^2$$

$$\operatorname{tg} \theta = \frac{y}{x}$$

Coordenadas cilíndricas



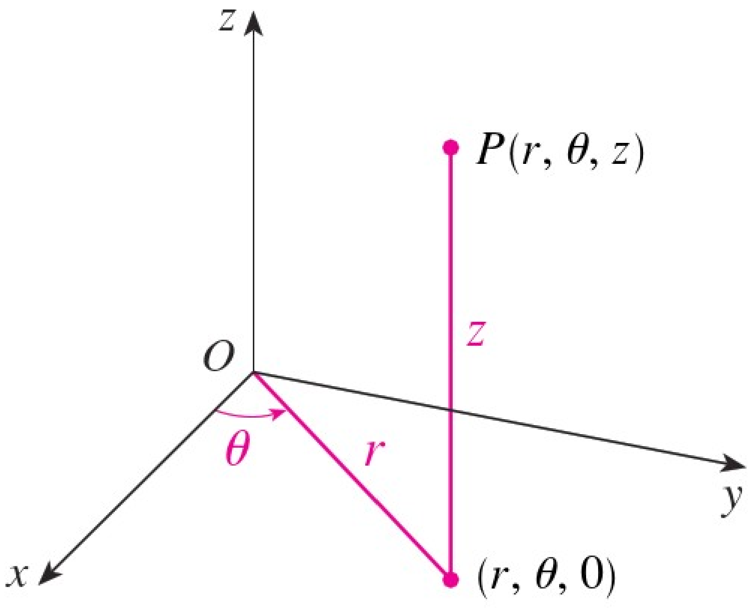
Coordenadas cilíndricas



$$x = r\cos\theta \qquad y = r\,\text{sen}\,\theta \qquad z = z$$

$$r^2 = x^2 + y^2 \qquad \text{tg}\,\theta = \frac{y}{x} \qquad z = z$$

Coordenadas cilíndricas



$$x = r\cos\theta \qquad y = r\,\text{sen}\,\theta \qquad z = z$$

$$r^2 = x^2 + y^2 \qquad \text{tg}\,\theta = \frac{y}{x} \qquad z = z$$

$$-r^2 = -x^2 - y^2$$

Coordenadas cilíndricas são úteis em problemas que envolvem simetria em torno de um eixo e o eixo z é escolhido de modo a coincidir com o eixo de simetria.

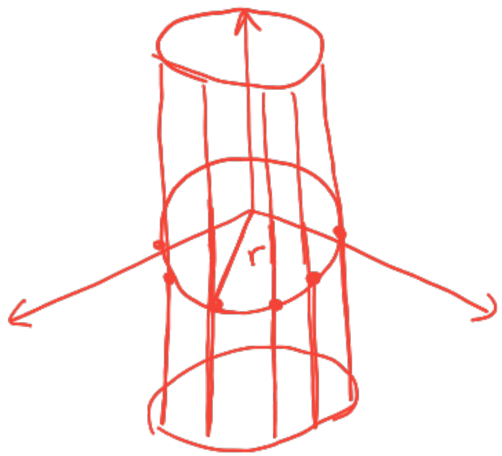
Exemplo

Descreva a superfície cuja equação em coordenadas cilíndricas é $r = c$.

$r = cte$

$0 \leq \theta \leq 2\pi$

$z \in \mathbb{R}$



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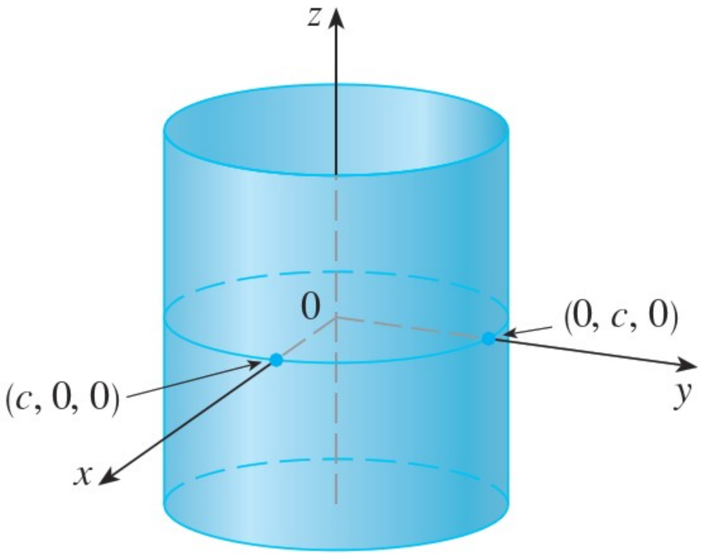


Exemplo

$(r \geq 0)$

$r = 1$

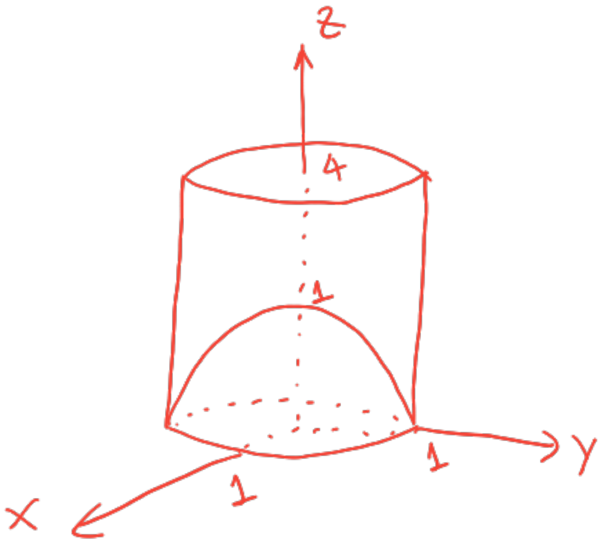
Um sólido E está contido no cilindro $x^2 + y^2 = 1$, abaixo do plano $z = 4$ e acima do paraboloide $z = 1 - x^2 - y^2$.

$z = 4$

$z = 1 - r^2$

$x^2 + y^2 = 1$

$x^2 + y^2 = 1 - z$ $\Rightarrow 1 = 1 - z \Rightarrow z = 0$



Exemplo

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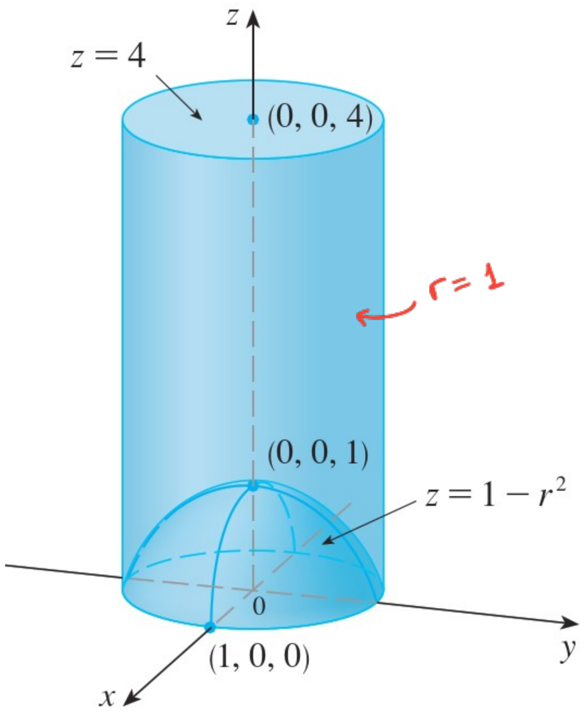
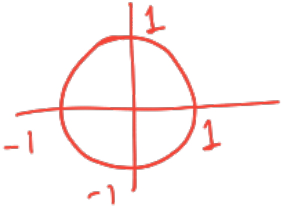
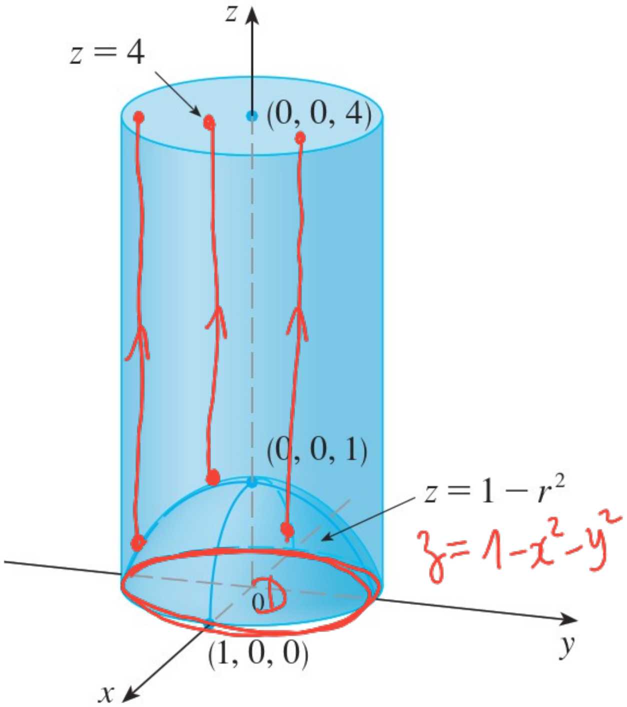


Exemplo



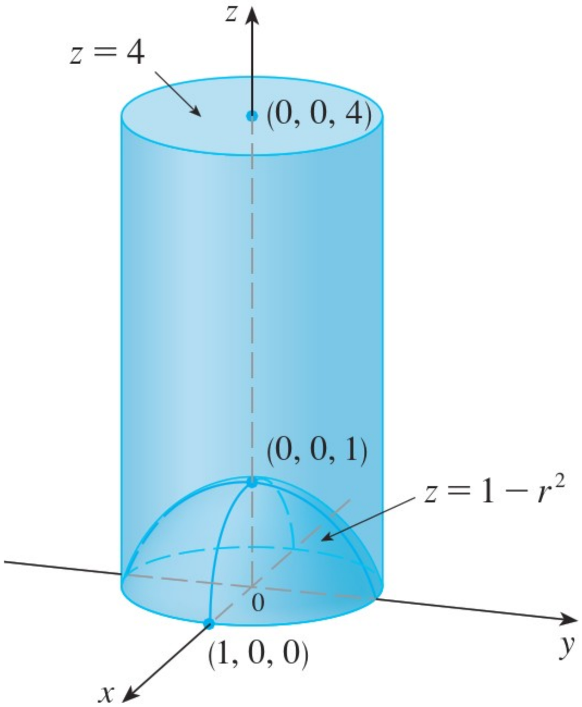


$$\iiint_E K\sqrt{x^2+y^2}\,dV = k\iint_D \left[\int_{1-x^2-y^2}^{4} \sqrt{x^2+y^2}\,dz\right] dA$$

$$= k\iint_D \sqrt{x^2+y^2}\cdot\left(z\Big|_{1-x^2-y^2}^{4}\right) dA = k\iint_D \sqrt{x^2+y^2}\cdot(3+x^2+y^2)\,dA$$

$$E = \{(x,y,z) \mid (x,y)\in D,\ 1-x^2-y^2\le z\le 4\} = \{(r,\theta,z) \mid 0\le r\le 1,\ 0\le\theta\le 2\pi,\ 1-r^2\le z\le 4\}$$

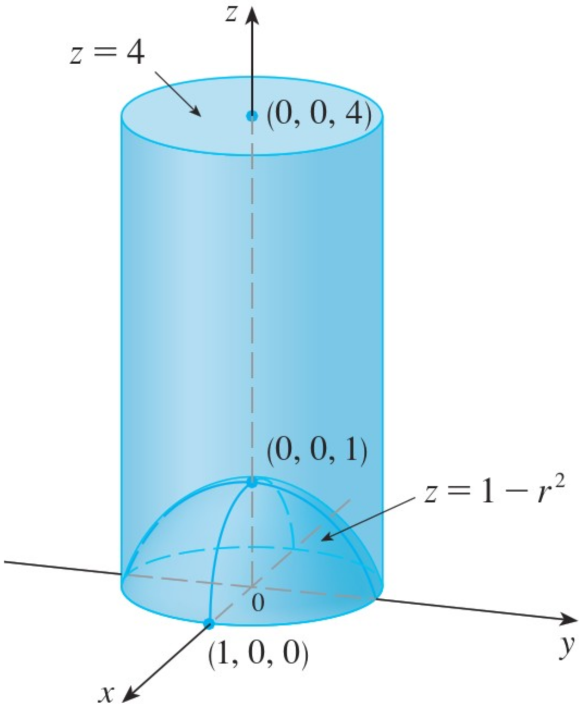
Exemplo



$$E = \left\{(r, \theta, z) \mid 0 \leqslant \theta \leqslant 2\pi,\ 0 \leqslant r \leqslant 1,\ 1 - r^2 \leqslant z \leqslant 4\right\}$$

$$\iiint_E K\sqrt{x^2 + y^2}\, dV$$

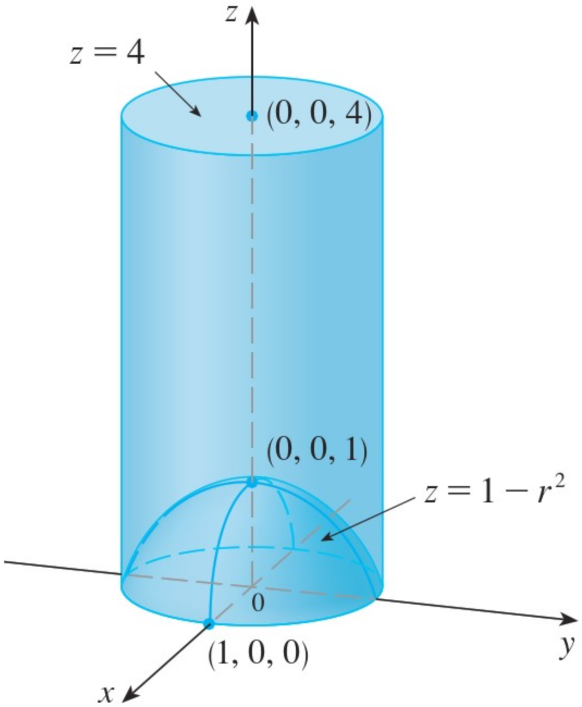
Exemplo



$$E = \{(r, \theta, z) \mid 0 \leqslant \theta \leqslant 2\pi,\ 0 \leqslant r \leqslant 1,\ 1 - r^2 \leqslant z \leqslant 4\}$$

$$\iiint_E K\sqrt{x^2 + y^2}\, dV = \int_0^{2\pi} \int_0^1 \int_{1-r^2}^4 (Kr)\, \boxed{r}\, dz\, dr\, d\theta$$

Exemplo



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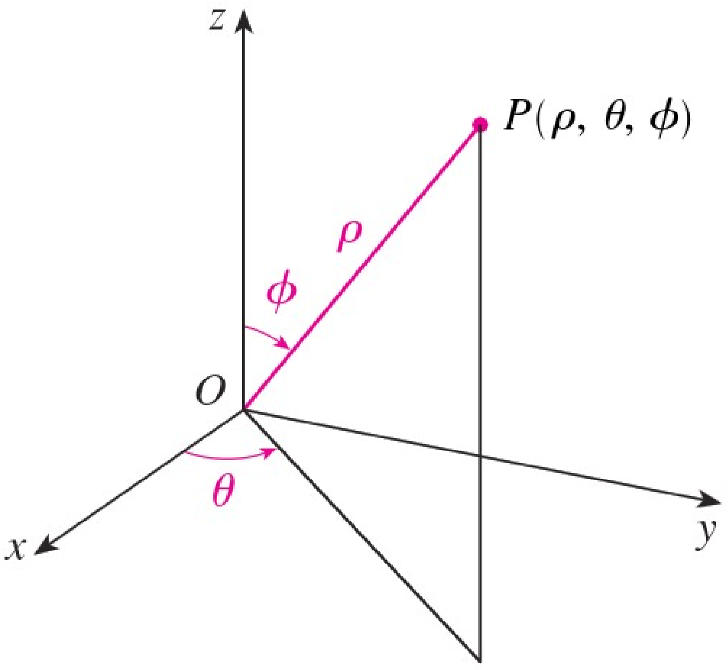
$$= \int_0^{2\pi} \int_0^1 Kr^2[4 - (1 - r^2)]\, dr\, d\theta$$

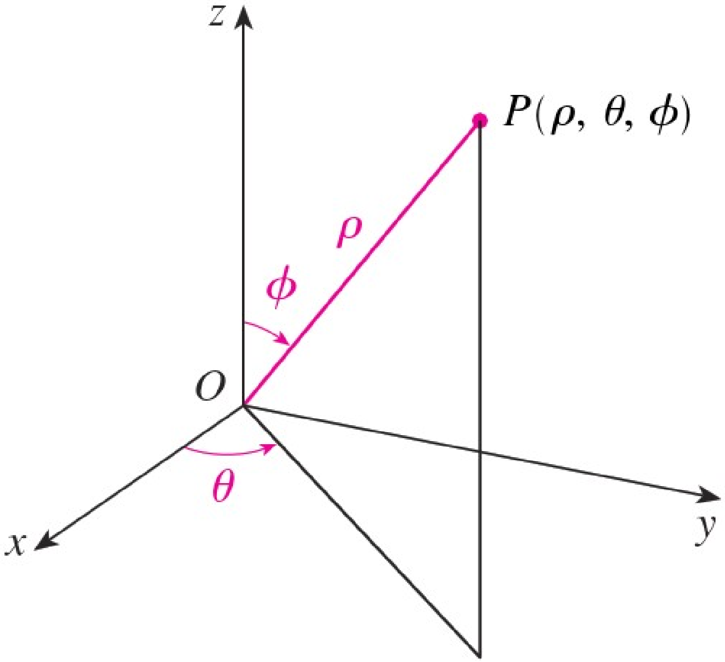
$$= K \int_0^{2\pi} d\theta \int_0^1 (3r^2 + r^4)\, dr$$

$$= 2\pi K\left[r^3 + \frac{r^5}{5}\right]_0^1 = \frac{12\pi K}{5}$$

Coordenadas esféricas



Coordenadas esféricas



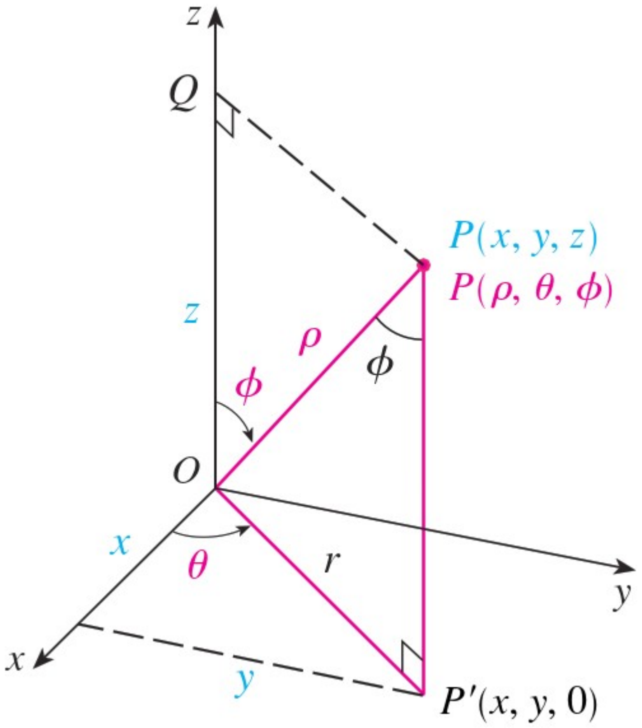
$\rho = |OP|$ é a distância da origem a P

θ é o mesmo ângulo que nas coordenadas cilíndricas

ϕ é o ângulo entre o eixo z positivo e o segmento de reta OP

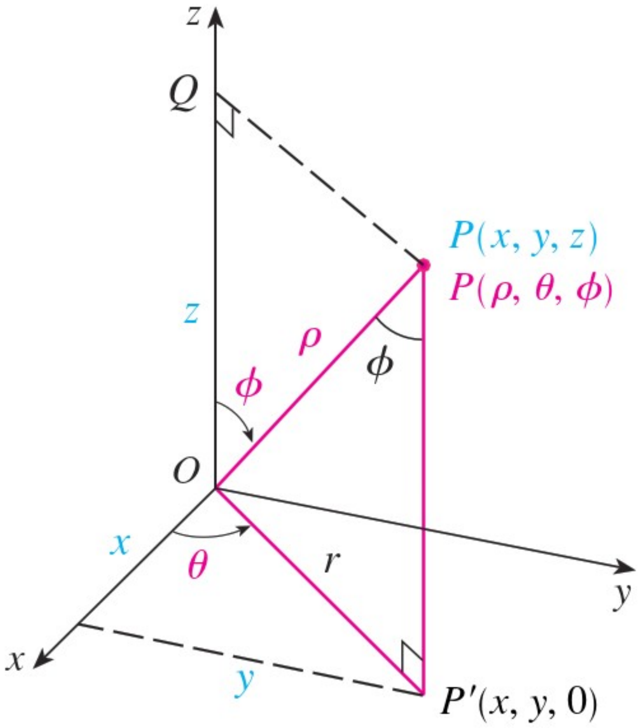
$$\rho \geqslant 0 \qquad 0 \leqslant \phi \leqslant \pi$$

Coordenadas esféricas



$$z = \rho \cos \phi \qquad r = \rho \operatorname{sen} \phi$$

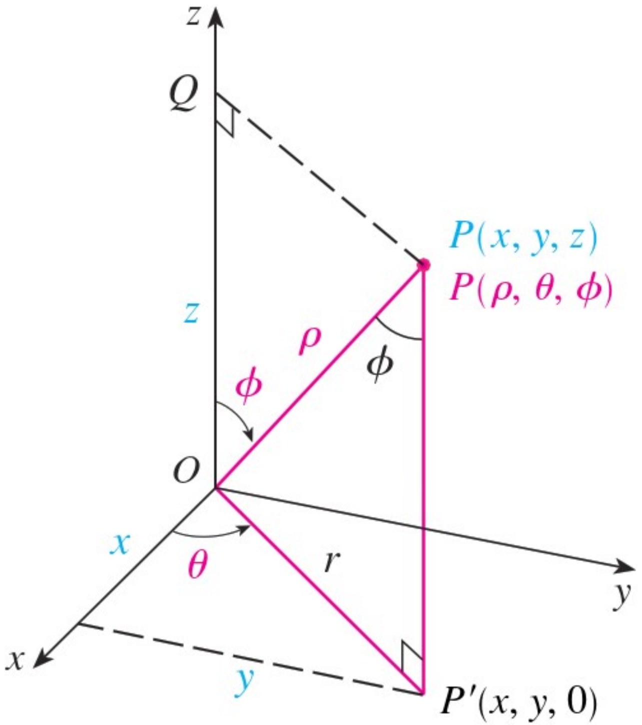
Coordenadas esféricas



$$z = \rho \cos \phi \qquad r = \rho \operatorname{sen} \phi$$

$$x = r \cos \theta \qquad y = r \operatorname{sen} \theta$$

Coordenadas esféricas



$$z = \rho \cos \phi \qquad r = \rho \operatorname{sen} \phi$$

$$x = r \cos \theta \qquad y = r \operatorname{sen} \theta$$

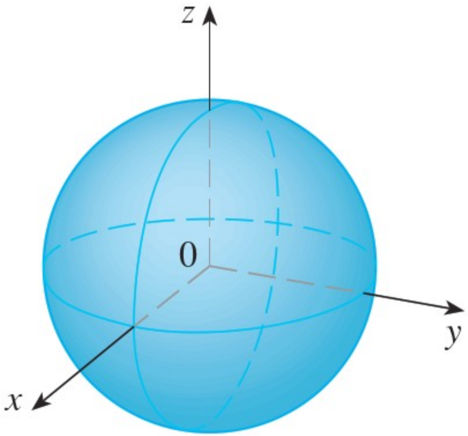
$$x = \rho \operatorname{sen} \phi \cos \theta$$

$$y = \rho \operatorname{sen} \phi \operatorname{sen} \theta$$

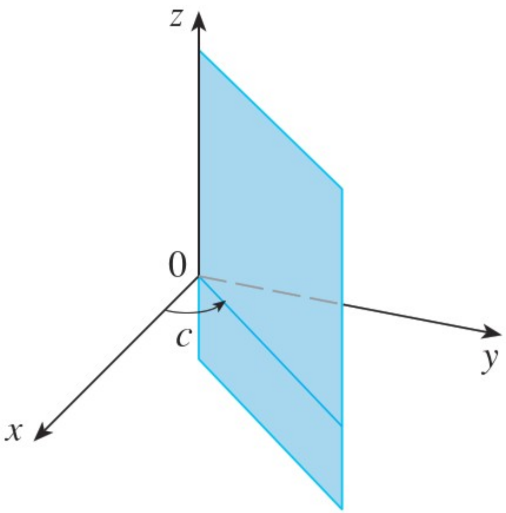
$$z = \rho \cos \phi$$

$$\rho^2 = x^2 + y^2 + z^2$$

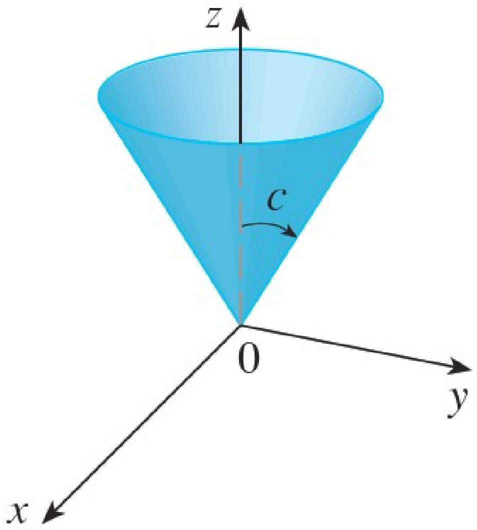
Exemplos



$\rho = c$, uma esfera



$\theta = c$, um semiplano



$0 < c < \pi/2$
$\phi = c$, um cone

Exemplo

Calcule $\iiint_B e^{(x^2+y^2+z^2)^{3/2}}\, dV$, onde B é a bola unitária:

$$B = \{(x, y, z) \mid x^2 + y^2 + z^2 \leqslant 1\}$$

$$= \{(\rho, \theta, \phi) \mid 0 \leq \rho \leq 1,\ 0 \leq \theta \leq 2\pi,\ 0 \leq \phi \leq \pi\}$$

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Calcule $\iiint_B e^{(x^2+y^2+z^2)^{3/2}}\, dV$, onde B é a bola unitária:

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$$\iiint_B e^{(x^2+y^2+z^2)^{3/2}}\, dV = \int_0^{\pi}\int_0^{2\pi}\int_0^{1} e^{(\rho^2)^{3/2}} \cdot \underbrace{\rho^2 \operatorname{sen}\phi}_{\text{Jacobiano}}\, d\rho\, d\theta\, d\phi$$

Exemplo

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$$\iiint_B e^{(x^2+y^2+z^2)^{3/2}}\, dV = \int_0^{\pi} \int_0^{2\pi} \int_0^1 e^{(\rho^2)^{3/2}} \boxed{\rho^2 \operatorname{sen} \phi}\ d\rho\, d\theta\, d\phi$$

$$u = \rho^3 \Rightarrow du = 3\rho^2 d\rho \to \rho^2 d\rho = \frac{1}{3} du$$

$$\int e^{\rho^3} \rho^2 d\rho = \int e^u \cdot \frac{1}{3} du = \frac{1}{3} e^u + c = \frac{1}{3} e^{\rho^3} + c$$

$$= \int_0^{\pi} \int_0^{2\pi} \int_0^1 e^{\rho^3} \cdot \rho^2 \operatorname{sen} \phi\, d\rho\, d\theta\, d\phi$$

$$= \int_0^1 e^{\rho^3} \cdot \rho^2 d\rho \cdot \int_0^{2\pi} d\theta \cdot \int_0^{\pi} \operatorname{sen} \phi\, d\phi$$

Exemplo

Calcule $\iiint_B e^{(x^2+y^2+z^2)^{3/2}}\, dV$, onde B é a bola unitária:

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$$B = \{(\rho, \theta, \phi) \mid 0 \leqslant \rho \leqslant 1,\ 0 \leqslant \theta \leqslant 2\pi,\ 0 \leqslant \phi \leqslant \pi\}$$

$$\begin{aligned}
\iiint_B e^{(x^2+y^2+z^2)^{3/2}}\, dV &= \int_0^{\pi} \int_0^{2\pi} \int_0^1 e^{(\rho^2)^{3/2}} \boxed{\rho^2 \operatorname{sen} \phi}\; d\rho\, d\theta\, d\phi \\
&= \int_0^{\pi} \operatorname{sen} \phi\, d\phi \int_0^{2\pi} d\theta \int_0^1 \rho^2 e^{\rho^3}\, d\rho \\
&= \left[-\cos \phi\right]_0^{\pi} (2\pi) \left[\tfrac{1}{3} e^{\rho^3}\right]_0^1 = \tfrac{4}{3}\pi(e - 1)
\end{aligned}$$

Exercícios

Utilize coordenadas esféricas para determinar o volume do sólido que fica acima do cone $z = \sqrt{x^2 + y^2}$ e abaixo da esfera $x^2 + y^2 + z^2 = z$.

$\frac{\pi}{8}$

$\rho^2 = \frac{1}{4} + \frac{1}{4}$, $\rho = \frac{\sqrt{2}}{2}$

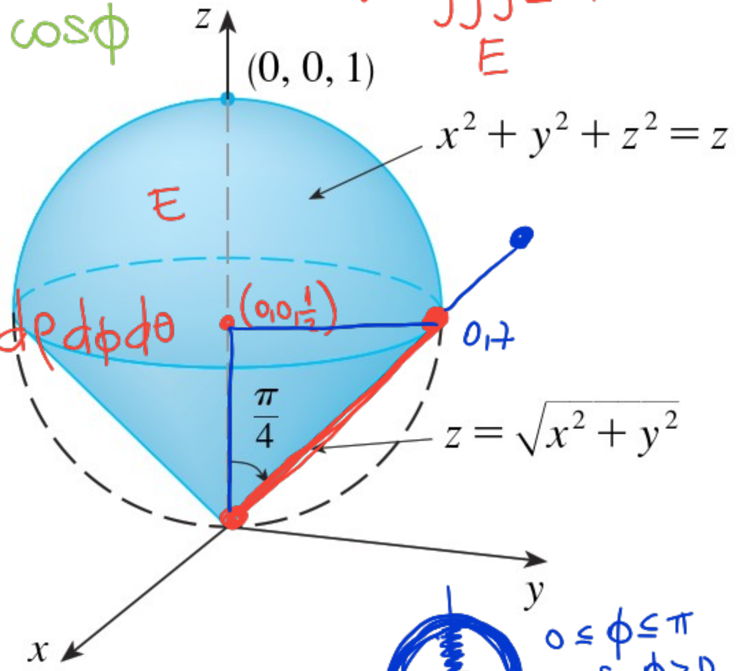
$0 \leq \theta \leq 2\pi$, $0 \leq \phi \leq \frac{\pi}{4}$, $0 \leq \rho \leq \cos\phi$

$\rho\cos\phi = \sqrt{\rho^2 \operatorname{sen}^2\phi\cos^2\theta + \rho^2\operatorname{sen}^2\phi\operatorname{sen}^2\theta} = \sqrt{\rho^2\operatorname{sen}^2\phi} = \rho\operatorname{sen}\phi$

$\Rightarrow \cos\phi = \operatorname{sen}\phi \Rightarrow \phi = \frac{\pi}{4}$.

$\rho^2 = \rho\cos\phi \Rightarrow \rho = \cos\phi$

$V = \iiint_E 1\, dV = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{\cos\phi} 1\cdot\rho^2\operatorname{sen}\phi\, d\rho\, d\phi\, d\theta$



Calcule $\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2} (x^2 + y^2)\, dz\, dy\, dx$.

$\frac{16}{5}\pi$

$E = \{(r, \theta, z) \mid 0 \leq r \leq 2,\ 0 \leq \theta \leq 2\pi,\ r \leq z \leq 2\}$

$0 \leq \phi \leq \pi$, $\operatorname{sen}\phi \geq 0$