$y' + P(x)\,y = Q(x)$ linear de 1ª ordem $\quad y(x)$

$\varphi = e^{\int P(x)\,dx}$ fator integrante

$\varphi y' + \varphi P(x)\,y = (\varphi y)'$

1) $(1+t)\dfrac{du}{dt} + u = 1+t\,, \quad t > -1 \quad u(t)$

$\div (1+t)$

$\Rightarrow \quad u' + \dfrac{1}{1+t}u = 1$

$\varphi = e^{\int \frac{1}{1+t}dt} = e^{\ln|1+t|} = |1+t| \overset{(t>-1)}{=} 1+t$

$\int \frac{1}{1+t}dt = \int \frac{1}{u}du$

$\left(\begin{matrix} u = 1+t \\ du = dt \end{matrix}\right) \quad = \ln|u| + c$

$= \ln|1+t| + c$

$\therefore (1+t)\left(u' + \dfrac{1}{1+t}u\right) = (1+t)\cdot 1 \Rightarrow (1+t)\,u' + u = 1+t$

$\Rightarrow \left[(1+t)\cdot u\right]' = 1+t \Rightarrow \int \left[(1+t)\,u\right]' dt = \int 1+t\,dt$

$\Rightarrow (1+t)\cdot u = \dfrac{t^2}{2} + t + c \Rightarrow u = \dfrac{t^2+2t+k}{2(1+t)} = \dfrac{t^2+2t+k}{2+2t}$

2) $y' + \dfrac{2x}{1+x^2}y = \dfrac{x^2}{1+x^2}\,, \quad y(0) = 2 \quad$ (linear)

$\varphi = e^{\int \frac{2x}{1+x^2}dx} = e^{\int \frac{1}{u}du} = e^{\ln|u|} = e^{\ln|1+x^2|} \overset{(x^2 \geq 0)}{=} |1+x^2| = 1+x^2$

$u = 1+x^2 \Rightarrow du = 2x\,dx$

$\therefore (1+x^2)\,y' + 2x\,y = x^2 \Rightarrow \left[(1+x^2)\cdot y\right]' = x^2$

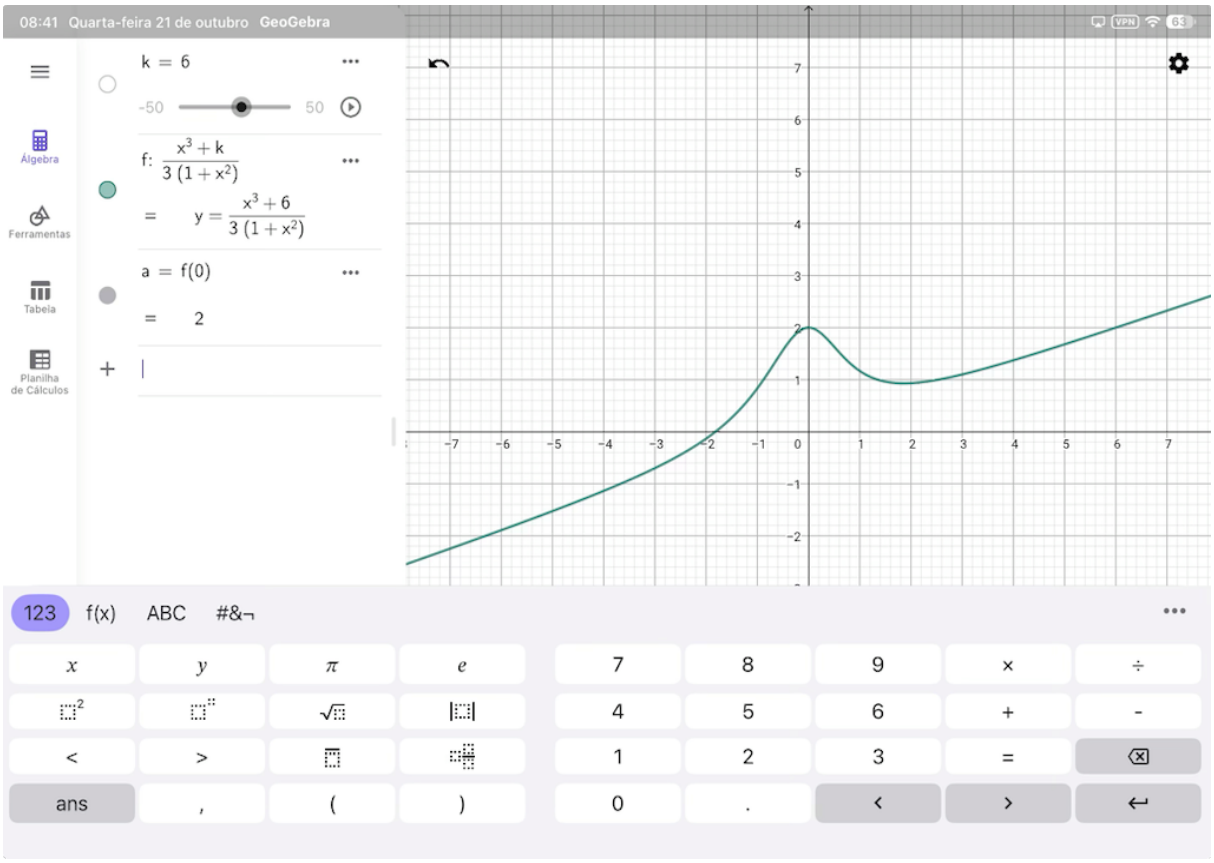
$\Rightarrow \int \left[(1+x^2)\cdot y\right]' dx = \int x^2 dx \Rightarrow (1+x^2)\cdot y = \dfrac{x^3}{3} + c \Rightarrow y = \dfrac{x^3+k}{3(1+x^2)}$

Problema de Valor Inicial (PVI): $y(0)=2$

$$y(x)=\frac{x^3+k}{3(1+x^2)}$$

$$y(0)=\frac{0^3+k}{3(1+0^2)}=\frac{k}{3} \quad \therefore \frac{k}{3}=2 \Rightarrow \boxed{k=6}$$

Portanto, $y=\frac{x^3+6}{3(1+x^2)}$



3) $P'=kP$, k constante $\qquad P(t)$

$P'-kP=0$ (linear), $\qquad \frac{1}{P}P'=k$ (separável)

$\varphi=e^{\int -k\,dt}=e^{-kt}$

$\int \frac{1}{P}\,dP=\int k\,dt$

$\therefore e^{-kt}P'-ke^{-kt}P=0$

$\Rightarrow \ln|P|=kt+c$

$\Rightarrow (e^{-kt}\cdot P)'=0$

$(P>0)$

$\Rightarrow \ln P=kt+c$

$$\Rightarrow \int (e^{-kt} \cdot P)' dt = \int 0 \, dt$$

$$\Rightarrow e^{-kt} P = C$$

$$\Rightarrow P = C e^{kt}$$

$$\Rightarrow e^{\ln P} = e^{kt + C}$$

$$\Rightarrow P = e^{kt} \cdot \underbrace{e^{C}}_{C_1} = C_1 e^{kt}$$