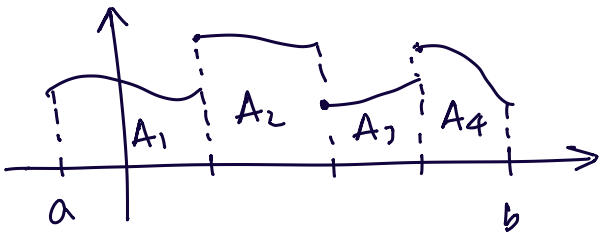
$$\int_a^b f(x)\,dx = F(b) - F(a), \quad \text{onde} \quad F'(x) = f(x)$$

$$F(x) = \int f(x)\,dx$$

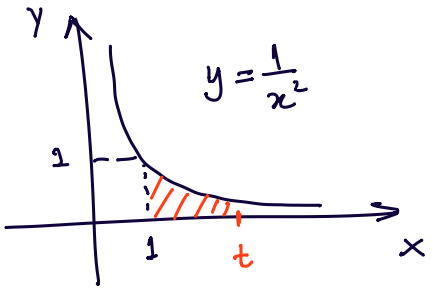
f precisa ser integrável



$$\int_a^b f(x)\,dx = A_1 + A_2 + A_3 + A_4$$

descontinuidades for finita

Integrais imprópria



$$A = \int_1^t \frac{1}{x^2}\,dx = \int_1^t x^{-2}\,dx = \left.\frac{x^{-1}}{-1}\right|_1^t$$

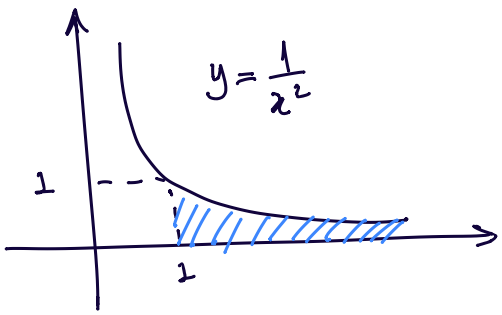
$$= -\left.\frac{1}{x}\right|_1^t = -\frac{1}{t} + 1 = \underbrace{1 - \frac{1}{t}}_{<1}, \quad \forall t \geq 1.$$

$t = 2$: $A = 1 - \frac{1}{2} = \frac{1}{2} = 0{,}5$

$t = 3$: $A = 1 - \frac{1}{3} = \frac{2}{3} \approx 0{,}66$

$t = 10$: $A = 1 - \frac{1}{10} = \frac{9}{10} = 0{,}9$

$\therefore \; t \to \infty \Rightarrow A \to 1$



$A = ?$

Definimos

$$A = \lim_{t \to \infty} A = \lim_{t \to \infty} \int_1^t \frac{1}{x^2}\,dx$$

Denotamos por $A = \int_1^{\infty} \frac{1}{x^2}\,dx$.

$$A = \lim_{t\to\infty} \int_1^t \frac{1}{x^2}\,dx = \lim_{t\to 1} 1 - \frac{1}{t} = 1$$

Definição:

1) Se $\int_a^t f(x)\,dx$ existe p/ todo $t \geq a$, então

$$\int_a^{\infty} f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$

se o limite existir.

2) Se $\int_t^b f(x)\,dx$ existe p/ todo $t \leq b$, então

$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx$$

se o limite existir.

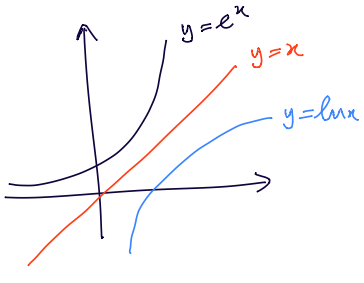
3) $$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^{\infty} f(x)\,dx$$

Exemplos: 1) $$\int_1^{\infty} \frac{1}{x}\,dx \overset{(\text{def.})}{=} \lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx$$

$$\int \frac{1}{x}\,dx \overset{(x>1)}{=} \ln x + c \quad \Rightarrow \int_1^t \frac{1}{x}\,dx = \ln t - \ln 1 = \ln t$$

$$\therefore \int_1^{\infty} \frac{1}{x}\,dx = \lim_{t\to\infty} \ln t = +\infty$$

A integral diverge.



$$2) \int_{-\infty}^{0} x e^x dx \overset{(\text{def.})}{=} \lim_{t \to -\infty} \int_{t}^{0} x e^x dx$$

$$\int x e^x dx$$

Integrando por partes:

$$\begin{aligned} u &= x & &\Rightarrow & du &= dx \\ dv &= e^x dx & & & v &= e^x \end{aligned}$$

$$\therefore \int x e^x dx = x e^x - \int e^x dx = x e^x - e^x + c$$

$$\Rightarrow \int_{t}^{0} x e^x dx = x e^x - e^x \Big|_{t}^{0} = 0 \cdot e^0 - e^0 - t e^t + e^t = -1 - t e^t + e^t$$

Assim,

$$\int_{-\infty}^{0} x e^x dx = \lim_{t \to -\infty} \left(-1 - t e^t + e^t \right)$$

Estudando os limites:

- $\lim_{t \to -\infty} -1 = -1$
- $\lim_{t \to -\infty} e^t = 0$
- $\lim_{t \to -\infty} -t e^t$ (indeterminação tipo $0 \cdot \infty$)

$$-t \cdot e^t = \begin{cases} \dfrac{-t}{e^{-t}} \to \dfrac{\infty}{\infty} \\ \dfrac{e^t}{-\frac{1}{t}} \to \dfrac{0}{0} \end{cases} \quad \text{ou}$$

$$-t \to \infty, \quad e^{-t} \to \infty, \quad e^t \to 0, \quad -\frac{1}{t} \to 0$$

$$= \lim_{t \to -\infty} \frac{-t}{e^{-t}} \overset{(\text{L'Hop.})}{=} \lim_{t \to -\infty} \frac{-1}{-e^{-t}} = \lim_{t \to -\infty} e^t = 0.$$

Portanto,

$$\int_{-\infty}^{0} x e^x \, dx = \lim_{t \to -\infty} -1 + \lim_{t \to -\infty} -te^t + \lim_{t \to -\infty} e^t$$

$$= -1 + 0 + 0 = -1 \quad \text{(converge)}$$

$x < 0$: $\underbrace{x}_{<0} \cdot \underbrace{e^x}_{>0} < 0$ gráfico está abaixo do eixo x.

3) $\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx \overset{(def.)}{=} \int_{-\infty}^{0} \frac{1}{1+x^2} dx + \int_{0}^{\infty} \frac{1}{1+x^2} dx$

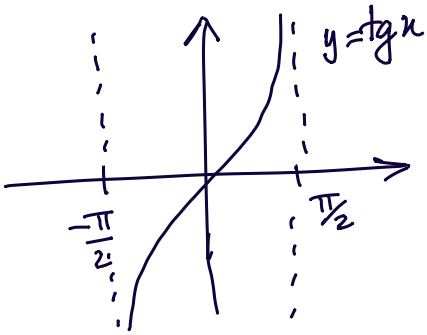
$$\left(\operatorname{arctg} x\right)' = \frac{1}{1+x^2}$$

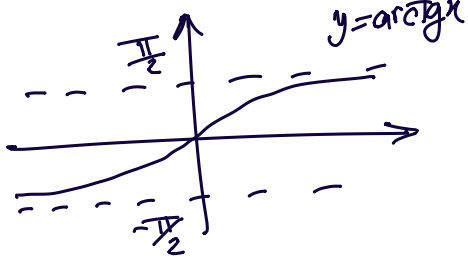
$$\int_{-\infty}^{0} \frac{1}{1+x^2} dx = \lim_{t \to -\infty} \int_{t}^{0} \frac{1}{1+x^2} dx = \lim_{t \to -\infty} \operatorname{arctg} x \Big|_{t}^{0} = \lim_{t \to -\infty} \operatorname{arctg} 0 - \operatorname{arctg} t$$

$$= \lim_{t \to -\infty} -\operatorname{arctg} t = \frac{\pi}{2}$$

$$\int_{0}^{\infty} \frac{1}{1+x^2} dx = \lim_{t \to \infty} \int_{0}^{t} \frac{1}{1+x^2} dx = \lim_{t \to \infty} \operatorname{arctg} x \Big|_{0}^{t} = \lim_{t \to \infty} \operatorname{arctg} t - \operatorname{arctg} 0$$

$$= \lim_{t \to \infty} \operatorname{arctg} t = \frac{\pi}{2}$$





Portanto, $\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx = \frac{\pi}{2} + \frac{\pi}{2} = \pi$

Exercício: $\int \frac{1}{1+x^2}\,dx = \operatorname{arctg} x + c \qquad (\operatorname{tg}\theta = x)$